Benjamin Franklin Finkel

BENJAMIN FRANKLIN FINKEL

Professor Benjamin Franklin Finkel, founder of the AMERICAN MATHEMATICAL MONTHLY, was born July 5, 1865, and died February 5, 1947. He studied in the rural schools of Ohio and taught in the schools of Ohio and Tennessee, with short periods of study at a normal school, until he became an instructor in mathematics and astronomy at Kidder Institute, Kidder, Missouri, in 1892. During these years he had become keenly aware of the poor instruction given in elementary mathematics, and he had indulged his natural bent for solving problems by contributing solutions to several mathematical and educational journals. While at Kidder Institute he published his well-known *Mathematical Solution Book*, and proposed to establish a journal "devoted solely to mathematics and suitable to the needs of teachers of mathematics in these schools" (high schools and academies).

In 1895, he was made Professor of Mathematics and Physics in Drury College, Springfield, Missouri, and continued in that position, with periods of study at the University of Chicago and the University of Pennsylvania, until his retirement in 1937. After his retirement he published a history of American mathematical journals in 1940, and he taught army classes in the University of Missouri during 1944. He received the degrees of A.M. and Ph.D. from the University of Pennsylvania in 1904 and 1906, respectively, and was honored with the degree of LL.D. by Drury College in 1923.

The first number of the MONTHLY appeared in January, 1894, and Professor Finkel continued to bear the sole responsibility for the editorship and the business management of the journal until he was joined by L. E. Dickson in October, 1902, by H. E. Slaught in January, 1907, and by G. A. Miller in January, 1909. The MONTHLY was formally transferred in January, 1913, to a board of editors representing fourteen colleges and universities. Then, in January, 1916, the magazine became the official journal of the Mathematical Association of America. Professor Finkel remained one of the editors of the department of *Problems and Solutions* through 1933, and was still a member of the board of editors at the time of his death.

The present writer has known perhaps thirty individuals who are the fortunate possessors of a full set of the volumes of the MONTHLY; thirty-five libaries have the full set, according to the Union List of Periodicals, and a few other libraries have the first volumes. Most of these early issues are accessible, so that those who wish to study the history of the journal may readily do so. A perusal of the first volumes will show the uneven character of the contents but, withal, a steady advance toward a journal of high grade. Quite early it was evident from the list of subscribers that there was little response from high school instructors and that the more fertile field was to be found in the colleges and universities.

The very first issue contained an article by a young graduate student in the University of Texas, who was later to be known as Professor L. E. Dickson. In the first three volumes, to go no further, appeared contributions by such well-

known mathematicians as R. J. Aley, Fletcher Durell, Arnold Emch, George Bruce Halsted, Alexander Macfarlane, Artemas Martin, G. A. Miller, E. H. Moore, R. E. Moritz, I. J. Schwatt, D. E. Smith, and B. F. Yanney.

During the first fifteen or twenty years of the MONTHLY's existence there were great financial and technical obstacles; a picture of these difficulties was vividly presented by Professor Finkel in an address given at the meeting of the Association in Cleveland, January 1, 1931, and published in the MONTHLY for June-July of that year. American mathematics is indebted to him for his imagination and his persistent courage in establishing and promoting a journal which has become most important in the field of mathematics. His zeal, intelligently directed as it was along useful lines, has earned for him a unique place in the history of mathematical publications.

WILLIAM DEWEESE CAIRNS

THE NEW SECRETARY-TREASURER

At the end of the present calendar year the term of office of our Secretary-Treasurer, Professor Walter B. Carver, will come to an end. Professor Carver accepted this position during the difficult period of the war with the understanding that he would be relieved at the end of his five-year term. After his long service to the Association, which includes five years as Editor-in-Chief and two years as President, he has well merited a respite from its labors.

On behalf of the officers and Board of Governors of the Mathematical Association of America I am happy to announce the election of Professor Harry M. Gehman, Chairman of the Department of Mathematics of the University of Buffalo, as Secretary-Treasurer for the five-year term beginning January 1, 1948.

LESTER R. FORD, *President*

REMARKS ON ANALYTICITY AND INTEGRATION

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1. Introduction. The theory of functions of a complex variable makes implicit use of Stieltjes integration in defining the concept of the integral of a continuous, complex-valued function along a rectifiable curve. To be more precise, suppose that $f(z)$ is a complex-valued function, defined and continuous at the points $z=z(t)=x(t)+iy(t)$, where $x(t)$ and $y(t)$ are two real-valued continuous functions for $a\leqq t\leqq b$. If $u(x, y)$ and $v(x, y)$ are the real and imaginary parts of $f(z)$, let $\bar{u}(t)=u[x(t), y(t)]$ and $\bar{v}(t)=v[x(t), y(t)]$, $a\leqq t\leqq b$. Then the integral of $f(z)$ along $z(t)$, $a\leqq t\leqq b$, written $\int_{z(t)}f(z)dz$, is simply $\int_a^b\bar{u}(t)dx(t)-\int_a^b\bar{v}(t)dy(t)+i\left[\int_a^b\bar{v}(t)dx(t)+\int_a^b\bar{u}(t)dy(t)\right]$, these four terms being Stieltjes integrals. Consequently, the functions $x(t)$ and $y(t)$ are normally assumed to be of bounded variation which is the same thing as assuming that the curve $z=z(t)$, $a\leqq t\leqq b$ is rectifiable. It is true that Stieltjes integration is rarely mentioned explicitly in the development; but be that as it may, the condition that the curve is rectifiable is invariably imposed.

In contradistinction to this approach, it is convenient to record the usual process employed. A *partition*, π, of $a\leqq t\leqq b$, is a sequence of numbers $t_0, \tau_0, t_1, \cdots, t_{i-1}, \tau_i, t_i, \cdots, t_{k-1}, \tau_k, t_k$ such that: (*i*) $a=t_0<t_1<\cdots<t_{i-1}<t_i<\cdots t_{k-1}<t_k=b$, and (*ii*) $t_{i-1}\leqq\tau_i\leqq t_i$ for $i=1, \cdots, k$. The symbol $|\pi|$ is used to indicate the maximum of the numbers (t_i-t_{i-1}) for $i=1, \cdots, k$. The complex number $J[\pi, z(t), f]$ is defined to be $\sum_{i=1}^{k}[z(\tau_i)][z(t_i)-z(t_{i-1})]$. The usual procedure is to say that if $\lim_{|\pi|\to 0}J[\pi, z(t), f]$ exists, then this limit is defined to be $\int_{z(t)}f(z)dz$, and is called the integral of $f(z)$ along the curve $z=z(t)$, $a\leqq t\leqq b$. The assumption that the curve is rectifiable simply guarantees the existence of $\lim_{|\pi|\to 0}J[\pi, z(t), f]$.

It is to be noticed that the remarks to this point merely call for the definition of $f(z)$ along the curve $z=z(t)$. Suppose that $f(z)$ is defined and *continuous* for $z\epsilon G$, where G is a region (=connected open set) on the complex z-plane. Then, of course, as soon as a curve $z=z(t)$, $a\leqq t\leqq b$, is given with the property that: (*i*) $z(t)\epsilon G$, $a\leqq t\leqq b$, (*ii*) $z(t)$ is rectifiable, then the integral of $f(z)$ along the curve $z=z(t)$ exists. Moreover, if $f(z)$ is *analytic* in G, while $z=z_n(t)$, $a\leqq t\leqq b$, $n=0, 1, 2, \cdots$, is a sequence of curves, each having the above pair of properties, and $z_n(t)\rightrightarrows z_0(t)$, $a\leqq t\leqq b$, $(=z_n(t)$ converges uniformly to $z_0(t)$ for $a\leqq t\leqq b)$ then $\lim\int_{z_n(t)}f(z)dz=\int_{z_0(t)}f(z)dz$. This classical result shows that the integral is, so to speak, a continuous functional of the curve of integration.

This note was initiated by an interest in this continuity property, and is an attempt to analyze several details of the situation. For example, it is shown that, relative to the continuity property, the requirement that the limit curve be rectifiable is not necessary. That is, $z_0(t)$ need not have property (*ii*) above, and it is nevertheless true that the sequence of complex numbers $\int_{z_n(t)}f(z)dz$ converges. One can no longer say, of course, that the limit of this sequence of

complex numbers is $\int_{z_0(t)} f(z)dz$, inasmuch as this term is normally undefined unless $z_0(t)$ is rectifiable.

This continuity result is certainly not startling and the proof is recorded here merely for the sake of completeness. On the other hand, it is shown that the continuity property is not only necessary but also sufficient for the analyticity of $f(z)$. In fact stronger theorems are proved, in that several weaker conditions are shown to guarantee the analyticity of $f(z)$.

It will become apparent in the proofs that the necessity argument makes active use of the Cauchy Theorem, while for sufficiency, the Morera Theorem supplies the key. From one point of view, therefore, these results may be described as generalized Cauchy-Morera Theorems.

2. Preliminary concepts. It will not have escaped attention that the term *curve* has been used as synonymous with the term *continuous function* $z=z(t)$, $a \leqq t \leqq b$. It is often convenient to use an equivalence relation in the family of continuous functions and consider a curve as an equivalence class. Such a procedure would, in fact, be rather convenient in some of the proofs, and has not been employed here merely because the standard texts on the subject do not explicitly employ this device.

In this paper, therefore, a *curve* C is simply a *continuous function* $z=z(t)$, $a \leqq t \leqq b$. The point $z(a)$ is said to be the *first* point, while $z(b)$ is known as the *last* point of C. A second continuous function $z=\bar{z}(t)$, $a \leqq t \leqq b$ is another curve $\overline{C}$, distinct from C, unless $\bar{z}(t)=z(t)$, $a \leqq t \leqq b$. The curve C: $z=z(t)$, $a \leqq t \leqq b$, is said to be *contained in a point set* G, if $z(t)$ is in G for every t in the interval $a \leqq t \leqq b$. (Notation: $C \subset G$ or $G \supset C$.) In the event a curve C: $z=z(t)$, $a \leqq t \leqq b$, has the property that $z(t)$ is a constant, ζ, for every value of t in the interval $a \leqq t \leqq b$, then ζ is sometimes used to denote the curve C. A sequence of curves, $\{C_n\}$, is said to converge to a curve C_0 if C_n is a continuous function $z=z_n(t)$ defined on the same interval $a \leqq t \leqq b$ for every n, and $z_n(t) \rightrightarrows z_0(t)$, $a \leqq t \leqq b$, where C_0 is the continuous function $z_0(t)$, $a \leqq t \leqq b$. (Notation: $C_n \to C_0$.) A curve C: $z=z(t)$, $a \leqq t \leqq b$, is said to be *closed* if $z(a)=z(b)$, that is, if the first and last points are the same.

It is convenient to have a specific notation for *rectifiable* curves and these are designated by the German letter $\mathfrak{C}$, in contradistinction to C which stands for a *general curve*.

For convenience in printing, if $f(z)$ is defined and continuous in a region G, while $\mathfrak{C}$: $z=z(t)$, $a \leqq t \leqq b$, is a rectifiable curve in G, then $I(\mathfrak{C}, f)$ will replace $\int_{z(t)} f(t)dz$. In fact, if the function $f(z)$ is fixed in the course of an argument, then the above notation will be abbreviated to $I(\mathfrak{C})$. As usual, the symbol $|I(\mathfrak{C})|$ stands for the absolute value of the complex number $I(\mathfrak{C})$. In accordance with the usual convention, the symbol $I(C)$ has meaning only for rectifiable curves C.

3. The theorems. A complex-valued function, $f(z)$, defined and continuous for z in a region G is said to have property:

$\mathcal{A}$ If it is analytic in G, that is, if it has a derivative at every point of G.

$\mathcal{C}$ If $\mathfrak{C}_n \to C_0$ and $C_0 \subset G \supset \mathfrak{C}_n$, $n=1, 2, 3, \cdots$, imply that $\lim I(\mathfrak{C}_n)$ exists.

$\mathcal{Z}$ If $\mathfrak{C}_n \to \zeta$ and $\zeta \subset G \supset \mathfrak{C}_n$, $n=1, 2, 3, \cdots$, imply that $\lim I(\mathfrak{C}_n)=0$.

$\mathcal{B}$ If $\mathfrak{C}_n \to \zeta$ and $\zeta \subset G \supset \mathfrak{C}_n$, $n=1, 2, 3, \cdots$, imply that there is a real number B associated with the sequence $\{\mathfrak{C}_n\}$ such that $|I(\mathfrak{C}_n)| < B$, $n=1, 2, 3, \cdots$.

It will be shown that these four properties are equivalent.

THEOREM 1. *$\mathcal{A}$ implies $\mathcal{C}$.*

Proof. Since $\mathfrak{C}_n \to C_0$, each $\mathfrak{C}_n$ is a function $z=z_n(t)$ on an interval $a \leqq t \leqq b$, while $z_n(t) \rightrightarrows z_0(t)$, $a \leqq t \leqq b$, and C_0 is the function $z_0(t)$, $a \leqq t \leqq b$. The set of points given by the formula $z=z_0(t)$, $a \leqq t \leqq b$, is closed, and as $C_0 \subset G$ there is a $\delta > 0$ such that the set of points, z, determined by the inequality $|z-z_0(t)| < \delta$ lies in G for $a \leqq t \leqq b$. Moreover the function $z_0(t)$, $a \leqq t \leqq b$, is continuous on a closed interval, and hence is uniformly continuous. Consequently, there is a sequence of t's, $a=t_0<t_1< \cdots <t_{k-1}<t_k=b$, such that $|z_0(t)-z_0(t_i)| < \delta/2$ for $t_{i-1} \leqq t \leqq t_i$, $i=1, \cdots, k$.

Let R_i be the set of points, z, determined by the inequality $|z-z_0(t_i)| < \delta$, $i=1, \cdots, k$. Then $R_i \subset G$. Define $\mathfrak{C}_n^i$ to be the curve $z=z_n(t)$, $t_{i-1} \leqq t \leqq t_i$; $i=1, \cdots, k$; $n=1, 2, 3, \cdots$. (The use of the German letter is justified because this curve is rectifiable.)

The curve $\mathfrak{A}_{np}^i: z=z_n(t_i)+[z_{n+p}(t_i)-z_n(t_i)]t$, $0 \leqq t \leqq 1$; $n=1, 2, 3, \cdots$; $p>0$; $i=0, \cdots, k$, is sometimes called the directed segment from $z_n(t_i)$ to $z_{n+p}(t_i)$. It is certainly rectifiable but need not lie in G.

On the other hand, since $z_n(t) \rightrightarrows z_0(t)$, $a \leqq t \leqq b$, there is an n_0 such that for $n > n_0$,

$$|z_n(t) - z_0(t)| < \delta/2, \qquad a \leqq t \leqq b.$$

Hence for $n > n_0$,

$$\mathfrak{C}_n^i \subset R_i, \qquad i = 1, \cdots, k, \text{ and}$$

$$\mathfrak{A}_{np}^{i-1} \subset R_i \supset \mathfrak{A}_{np}^i \qquad i = 1, \cdots, k.$$

But $f(z)$ is analytic in R_i, a simply connected region, for $i=1, \cdots, k$; hence the Cauchy Theorem implies that there is an analytic function $F_i(z)$, in R_i, $i=1, \cdots, k$, such that if $\mathfrak{C}: z=z(t)$, $c \leqq t \leqq d$ is a rectifiable curve in R_i, then $I(\mathfrak{C})=F_i[z(d)]-F_i[z(c)]$.

Therefore, $I(\mathfrak{C}_n^i) + I(\mathfrak{A}_{np}^i) - I(\mathfrak{C}_{n+p}^i) - I(\mathfrak{A}_{np}^{i-1}) = F_i[z_n(t_i)] - F_i[z_r(t_{i-1})] + F_i[z_{n+p}(t_i)] - F_i[z_n(t_i)] - F_i[z_{n+p}(t_i)] + F_i[z_{n+p}(t_{i-1})] - F_i[z_{n+p}(t_{i-1})] + F_i[z_n(t_{i-1})] = 0$.

(This device is employed because the symbol $(\mathfrak{C}_n^i + \mathfrak{A}_{np}^i - \mathfrak{C}_{n+p}^i - \mathfrak{A}_{np}^{i-1})$ has not been defined. Without using an equivalence relation it is not possible to define such a formal sum as a closed *curve* and use the Cauchy Theorem directly. An alternative device is to define the above symbol as a *chain* and consider integration over chains. In this connection the reader may consult the elegant development of Artin [1].)

To return to the argument, $\sum_{i=1}^{k} I(\mathfrak{C}_n^i) - \sum_{i=1}^{k} I(\mathfrak{C}_{n+p}^i) = I(\mathfrak{A}_{np}^0) - I(\mathfrak{A}_{np}^k)$; but $\sum_{i=1}^{k} I(\mathfrak{C}_n^i) = I(\mathfrak{C}_n)$, for any n, hence

$$|I(\mathfrak{C}_n) - I(\mathfrak{C}_{n+p})| \leqq |I(\mathfrak{A}_{np}^0)| + |I(\mathfrak{A}_{np}^k)| \leqq |F_1[z_n(a)] - F_1[z_{n+p}(a)]| + |F_k[z_n(b)] - F_k[z_{n+p}(b)]|.$$

The continuity of F_1 and F_k together with the fact that $\lim z_n(t) = z(t)$ for $t = a$ and b, guarantees that, for any $\epsilon > 0$ there is an $n_1 > n_0$ such that if $n > n_1$ then

$$|I(\mathfrak{C}_n) - I(\mathfrak{C}_{n+p})| < \epsilon \qquad \text{for } p > 0.$$

Hence $\lim I(\mathfrak{C}_n)$ exists.

THEOREM 2. *$\mathcal{C}$ implies $\mathcal{Z}$.*

Proof. Since $\mathfrak{C}_n \to \zeta$ it is clear that the sequence $\mathfrak{C}_1, \zeta, \mathfrak{C}_2, \zeta, \cdots \to \zeta$. Hence the sequence $I(\mathfrak{C}_1), I(\zeta), I(\mathfrak{C}_2), I(\zeta), \cdots$ converges. But $I(\zeta) = 0$, therefore the limit must be zero, showing that $\lim I(\mathfrak{C}_n) = 0$.

THEOREM 3. *$\mathcal{Z}$ implies $\mathcal{B}$.*

This is quite obvious.

THEOREM 4. *$\mathcal{B}$ implies $\mathcal{A}$.*

Proof. If $f(z)$ is not analytic in G, then it fails to be analytic at some point ζ in G. Suppose that the distance from ζ to the boundary of G is greater than $\delta > 0$. Let R_n be the set of z's determined by the unequality $|z - \zeta| < \delta/n$, $n = 1, 2, 3, \cdots$. For each n, $R_n \subset G$ and is simply connected. If $I(\mathfrak{C}) = 0$ for every *closed* curve $\mathfrak{C} \subset R_n$ then by the Morera Theorem, $f(z)$ is analytic in R_n. Hence there must be a closed curve $\mathfrak{C}_n$: $z = z_n(t)$, $a_n \leqq t \leqq b_n$ such that $I(\mathfrak{C}_n) \neq 0$, $n = 1, 2, 3, \cdots$. For each n there is an integer $k_n > 0$ such that $k_n \cdot |I(\mathfrak{C}_n)| > n$. Let $l_n^i(t)$ be the unique linear mapping such that $l_n^i[(i-1)/k_n] = a_n$ and $l_n^i(i/k_n) = b_n$; $n = 1, 2, 3, \cdots$; $i = 1, \cdots, k_n$. Consider the curve $\mathfrak{C}_n^i$: $z = z_n^i(t) = z_n[l_n^i(t)]$, $(i-1)/k_n \leqq t \leqq i/k_n$, it is easy to see that the integral of $f(z)$ along $\mathfrak{C}_n$ is equal to the integral of $f(z)$ along $\mathfrak{C}_n^i$, $n = 1, 2, 3, \cdots$; $i = 1, \cdots, k_n$.

For each n, let $\bar{z}_n(t)$ be the function defined on $0 \leqq t \leqq 1$ as equal to $z_n^i(t)$ for $(i-1)/k_n \leqq t \leqq i/k_n$, $i = 1, \cdots, k_n$. Then, as each $z_n(t)$ is closed, $\bar{z}_n(t)$ is a continuous function on $0 \leqq t \leqq 1$, and so is a curve $\bar{\mathfrak{C}}_n$. Moreover,

$$|I(\bar{\mathfrak{C}}_n)| = \left| \sum_{i=1}^{k_n} I(\mathfrak{C}_n^i) \right| = k_n |I(\mathfrak{C}_n)| > n, \quad n = 1, 2, 3, \cdots.$$

On the other hand $\bar{\mathfrak{C}}_n \subset R_n$ and hence $|\bar{z}_n(t) - \zeta| < \delta/n$, $0 \leqq t \leqq 1$. Consequently $\bar{\mathfrak{C}}_n \to \zeta$, and by hypothesis there is a number B such that $|I(\bar{\mathfrak{C}}_n)| < B$, $n = 1, 2, 3, \cdots$.

This contradiction shows that $f(z)$ must be analytic in G.

4. General comments. The first theorem makes it possible to define a complex number $I^*(C, f)$ for any curve, C, and a function, $f(z)$, analytic in a region, G, containing C. In fact, let $\{\mathfrak{C}_n\}$ be a sequence of rectifiable curves such that $\mathfrak{C}_n \to C$. Such sequences certainly exist: in particular each $\mathfrak{C}_n$ may be taken to be a polygon inscribed in C. Moreover, it may be assumed that $\mathfrak{C}_n \subset G$, $n=1, 2, 3, \cdots$. Let $I^*(C, f) = \lim I(\mathfrak{C}_n, f)$, which is known to exist by Theorem 1. In the event $\overline{\mathfrak{C}}_n \to C$, then the sequence $\mathfrak{C}_1, \overline{\mathfrak{C}}_1, \mathfrak{C}_2, \overline{\mathfrak{C}}_2, \cdots \to C$, from which it follows that $\lim I(\mathfrak{C}_n, f) = \lim I(\overline{\mathfrak{C}}_n, f)$. In other words $I^*(C, f)$ depends only on C and f. This number may be called *a continuous extension* of the ordinary integral, since it is but a remark to show that it agrees with the ordinary integral in the event C is rectifiable. Moreover, it enjoys all the common properties of the ordinary integral; in particular it is a continuous functional of the curve of integration. Indeed, it may be spoken of as *the continuous extended integral*, since any extension of the ordinary integral which is a continuous functional of the curve of integration must agree with this particular extension. In view of these comments it is possible to state—

THEOREM 5. *A function $f(z)$ which is continuous in a region G is analytic in G if and only if the continuous extended integral of $f(z)$ exists along every curve C in G.*

It should be pointed out that, if $f(z)$ is analytic in a simply connected region R, then the continuous extended integral of $f(z)$ along a curve C in R may be defined as a trivial consequence of classical results. For, if $f(z)$ is analytic in R then, as has been noted in Theorem 1, there is an analytic function $F(z)$ in R such that, if $\mathfrak{C}: z=z(t)$, $a \leqq t \leqq b$ is in R, then $I(\mathfrak{C}, f) = F[z(b)] - F[z(a)]$. Consequently, if $C: z=\bar{z}(t)$, $\bar{a} \leqq t \leqq \bar{b}$ is a general curve, then the continuous extended integral $I^*(C, f) = F[\bar{z}(\bar{b})] - F[\bar{z}(\bar{a})]$ and may be so defined, *a priori*. For a region G which is not simply connected this artifice will not always work, as in general no such function $F(z)$ exists. The method outlined in this paper is, in a sense, simply one device which extends this trivial remark to general regions.

A question of some interest is concerned with the following observation. Suppose G is a region and $\mathfrak{F}$ the totality of functions each of which is defined and analytic on G. If C is a curve lying in G, then what properties of C are necessary and sufficient for the usual process (noted in the introduction) to yield an integral of $f(z)$ along C for every $f(z)$ in $\mathfrak{F}$? (The condition that $\mathfrak{C}$ is rectifiable is sufficient for the purpose, but it is not necessary.) The question has been completely answered by Zorn [2].

Let $C: z=z(t)$, $a \leqq t \leqq b$ be any curve in G and $\pi: t_0, \tau_1, t_1, \cdots, t_{k-1}, \tau_k, t_k$ be a partition of the interval $a \leqq t \leqq b$. The symbol $\pi(C)$ is defined to be $\sum_{i=1}^{k} |z(t_i) - z(t_{i-1})|^2$. The symbols $|\pi|$ and $J[\pi, z(t), f]$ have been defined in the introduction. For temporary convenience, the curve C will be called *quasi-rectifiable* if $\lim_{|\pi| \to 0} \pi(C) = 0$.

The result of Zorn is that $\lim J[\pi, z(t), f]$ *exists if and only if C is quasi-rectifiable.* There are curves which are quasi-rectifiable without being rectifiable; on the other hand, not every curve is quasi-rectifiable.

In conclusion, it is noted that, if $f(z)$ is in $\mathfrak{F}$ and C is a quasi-rectifiable curve in G, then $\lim J[\pi, z(t), f] = I^*(C, f)$.

References

1. Artin, Emil. On the Theory of Complex Functions. Notre Dame Mathematical Lectures, No. 4, (1944) pp. 57–70.

2. Zorn, M. A. Approximating Sums. This MONTHLY, vol. 54, (1947) pp. 148–151.

COLIN MACLAURIN

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On June 14, 1946, honor was paid to the memory of Colin Maclaurin when members of the Edinburgh Mathematical Society laid a wreath beside his grave in Greyfriars Churchyard at Edinburgh on the occasion of the bicentenary of his death. At a special meeting of the Society in December, 1946, a vivid tribute was paid by Dr. Robert Schlapp, a former President of the Society, who described the life and work of Maclaurin. I was fortunate to be present to hear the address, and am much indebted to what I heard in what follows.

Colin Maclaurin was born at Kilmodan, Argyll, in February, 1698. He was one of three sons. John, the eldest, following in his father's steps, became a minister; he was a public spirited man of profound learning, and corresponded with Jonathan Edwards, the American metaphysician. The second son, Daniel, died young after having given signs of extraordinary genius. Colin was the youngest. His father died when Colin was six weeks old, and nine years later he lost his mother. The care of the children thereafter devolved entirely upon their uncle. At the age of eleven Colin followed his brother to the University of Glasgow where he came under the influence of Robert Simson, Professor of Mathematics, from whom he gained a lifelong interest in Euclid and ancient geometry. He graduated at the age of fifteen when he wrote and publicly defended a thesis, *On the power of gravity*. After spending a year reading Divinity he left Glasgow to live with his uncle in their highland home beside Loch Fyne, reading mathematics, philosophy, and the classics, wandering on the hills, actively searching out the scientific secrets of their stones and plants. Unfinished scraps in his notebooks reveal the sensitivity of his nature as he would sometimes break out into poetic rhapsody on the beauties of the scene and the perfections of its Author. Such was the background of the startlingly mature work upon the organic description of plane curves which he completed in 1719, at the age of twenty-one.

Maclaurin was the third and greatest of three scholars who, more than any others, made a critical and understanding study of the *Principia* of Newton, that had been published at Halley's prompting in 1687. David Gregory, astronomer and mathematician, was the first; Roger Cotes, the second, died in 1716 at the age of 34. Cotes helped Newton prepare the second edition of the *Prin-*

cipia, and Newton once wrote, "Had Cotes lived we might have known something." Cotes left a confused heap of manuscripts from which one particular geometrical theorem was extracted by his cousin Robert Smith (who later became Master of Trinity College, Cambridge) and sent to Maclaurin. It led to his second great work which has proved to be the foundation stone of the theory of plane cubic curves.

At the age of nineteen Maclaurin was appointed to the Chair of Mathematics in the Marischal College, Aberdeen, where honor was shown to his memory at a special lecture given in the College on February 4, 1947. He first met Sir Isaac Newton the next year in London, where he left his *Geometria Organica* to be printed, and was admitted a Fellow of the Royal Society. In 1722 he undertook a Continental journey as the tutor to the son of Lord Polwarth, and actually extended his absence from his Chair to three years. During the tour he was mathematically active, and for a thesis upon the percussion of bodies he was awarded a prize by the French Academy. Other awards, the same year, were made to Leonard Euler and Daniel Bernoulli. Maclaurin returned to Aberdeen after the sudden death of his pupil in France, and expressed his regret at his long absence without leave! In the following January his Chair was declared vacant, but meantime he had moved in November to Edinburgh where he deputized for the ageing Professor James Gregory, the nephew of his more famous namesake. Eventually and on the recommendation of Newton he succeeded to the Edinburgh Chair, which he occupied with great distinction. His classes were well attended, and his books on algebra and fluxions were models of lucid, careful elaboration from the very beginnings to the latest and deepest discoveries. Maclaurin laid the actuarial foundations that have borne fruit in the well known insurance societies of Edinburgh. He was the prime mover in the formation of a scientific gathering that sprang from the Medical Society, and soon grew to be the Royal Society of Edinburgh. In 1733, he married Anne, the daughter of Mr. Walter Stewart, Solicitor General for Scotland. Of their seven children, two sons, John and Colin, and three daughters, survived him.

Maclaurin was an able experimentalist. He proposed an observatory for Scotland, took an active part in improving the maps of Orkney and Shetland, and was prepared to embark on a voyage to discover a North Polar passage, should the opportunity occur. In 1745, when a Highland army marched on Edinburgh, Maclaurin took a leading part in rallying the townsfolk against the Jacobites and organizing the defenses of the city. Night and day he planned and supervised the hastily erected fortifications, a task of devotion which damaged his health. When the city was taken, he escaped and withdrew to England, reaching York where he became the guest of Thomas Herring, the Archbishop. "Here," wrote Maclaurin, "I live as happily as a man can do who is ignorant of the state of his family, and sees the ruin of his country." He returned the next year to Edinburgh, but the rigors of the journey broke his health, and in 1746 he died at the early age of forty-eight. Only a few hours before his death he was engaged in dictating a concluding passage for his work on the Philosophy of Newton; the

argument in favor of a future life contained in the last sentences is now well known; it proceeded from the lips of a dying man. After speaking of the beautiful scheme of nature and how imperfectly it can be apprehended in man's present state, "this," he says, "naturally leads us to consider our present state as only the dawn or beginning of our existence, and as a state of preparation or probation for further advancement."

The first mathematical work of Maclaurin was his *Geometria Organica sive Descriptio Linearum Curvarum Universalis* (London 1720), many of the properties therein contained being discovered while he was in his teens. The book not only bore the imprimatur of Sir Isaac Newton but took its rise from Newton's method of generating a conic or a cubic by rotating two angles of fixed sizes about their vertices S and C as fixed pivots.

If, as in Figure 1, Q, the intersection of two of the legs, lies on a fixed straight line AE, then P, the intersection of the other two legs, describes a conic which passes through S and C. Maclaurin proved this by expressing z, the ordinate of Q,

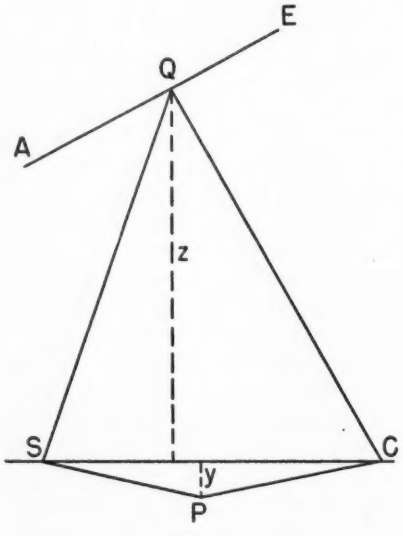

FIG. 1

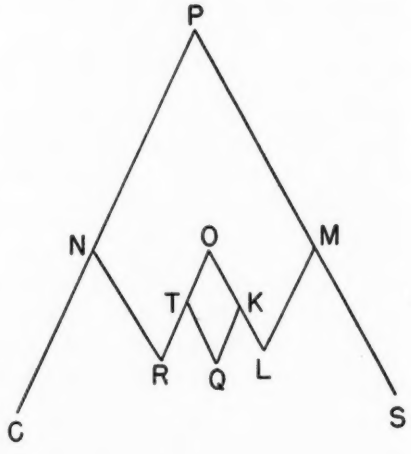

FIG. 2

bilinearly in terms of x and y, where C is the origin and CS is the axis of x, and again where S is the origin and SC the axis of x. The two values of z gave him a second degree equation in x and y. The free use of several origins within one geometrical figure, and the persistent use of the same parameters x and y to express all the necessary magnitudes, is typical of his work throughout. He advances step by step, always using his two pivots S and C until he reaches highly elaborate properties such as those involved in the consideration of Figure 2.

In Figure 2, let $CNRTQ$ be a succession of given angles CNR, NRT, and so

on, whose vertices N, R, T move on given lines (straight or curved) as guides, while C is a fixed point. Let $SMLKQ$ be a similar succession, where S is a fixed point. Let the first legs CN and SM reckoned from C and S meet at P, let the rth leg from S and the sth leg from C meet at O, while the mth from S and the nth from C meet at Q. (The figure, of course, requires $m=n=4$.) If Q moves on a guiding curve, then the point O describes a curve of order $lkitpuz\times(ms+nr+s+r)$ where the single letters denote the several orders of the curves described by Q, N, R, T, M, L, K. In particular, if all the guides are straight lines and $r=s=1$, the locus of P is a curve of order $m+n+2$.

A vast number of properties are worked out, and all the rudiments of higher plane curves are dealt with, their nodes and cusps, intersections and freedoms, the number $\frac{1}{2}(n-1)(n-2)$ for the maximum number of nodes on an n-ic, and the mn intersections of general curves of orders m and n. He printed a proof of this last important result in a sequel a few years later, but apparently did not publish it.

In 1722, Maclaurin learned from Robert Simson the beautiful property of Pappus, his *porism*, that if each of the three sides of a variable triangle is made to pivot on a point while two of the vertices describe straight lines, then the third vertex also will describe a straight line, provided that all three pivots are collinear. This suggested to Maclaurin a sequel to the above, by relaxing the constancy of the moving angles but, instead, constraining each leg (not merely the first and the last) to pivot on a fixed point. He obtained a formula analogous to the above, for the locus of O or P, and also found that the order of the locus of P was halved when all the pivots were collinear. From the triangle and three non-collinear pivots he obtained a variant of Pascal's Theorem, which he proved while he was at sea on his memorable journey to the Continent in July, 1722.

His second great geometrical work, the *De Linearum Geometricarum Proprietatibus Generalibus*, published as an appendix to his treatise on Algebra, after his death, was inspired by Cotes' Theorem, and laid the foundation for the general theory of cubic curves. The work shows what can happen when two distinct ideas are perfectly blended. One idea came from Newton, and one from Cotes. From Newton came the property that the ratio of $PA\cdot PB\cdot PC$ to $Pa\cdot Pb\cdot Pc$ is constant when two lines through a variable point P cut a curve of order n in the points A, B, C and a, b, c, respectively, while remaining parallel to two fixed lines. From Cotes came the property $\Sigma PA^{-1}=nPQ^{-1}$, where Q is the point at which any line through P cuts the polar line of P with regard to the curve. The latter formula suggested logarithmic differentiation of the former, whence by a beautiful argument Maclaurin arrived at the property that if the tangents at the points $A, \cdots$, on the variable chord through P cut a fixed chord at the points $K, \cdots$, and this chord cut the curve at $D, \cdots$, then

$$\Sigma PD^{-1} = \Sigma PK^{-1}.$$

By applying this result to the cubic, taking P at an inflexion, or at another point of the curve, or again elsewhere, he discovered an immense number of properties,

many being reminiscent of the harmonic properties of conics. Typical examples are that four tangents can be drawn from a point of the curve, three only if the point is an inflexion, in which case the points of contact are collinear, and that a chord through two inflexions necessarily passes through a third.

The Treatise of Fluxions (Edinburgh, 1742) is a masterpiece of reasoning, in which Maclaurin gave a systematic account of Newton's theory, set out in both geometrical and analytical form, with a wealth of applications and many discoveries. It was written in answer to an attack which Bishop Berkeley made in *The Analyst* against the use of infinitesimals. In point of rigor it is a worthy link between the ancient method of exhaustions and the subsequent work of Cauchy and of Weierstrass. Maclaurin extended Newton's work on the gravitation of spheres to that of ellipsoids, and incidentally introduced the idea of confocal conics. The power of the work moved Clairaut to abandon analysis and attack the problem of the figure of the earth by pure geometry. Lagrange who, after Maclaurin, took the next great step forward in the theory, by developing the idea of potential, described this passage of Maclaurin as "le chef d'oevre de geometrie qu'on peut comparer a tout ce qu' Archimedes nous a laisse de plus beau et de plus ingenieux" (*Mem. de l'Acad. de Berlin*, 1773). *The Treatise of Fluxions* contains a theoretical and practical discussion of infinite series, including his well known integral test for convergence, the Maclaurin series, which he states to be a consequence of Brook Taylor's series (1715), and the summation formula discovered independently and nearly contemporaneously by Euler and Maclaurin. Perhaps the finest part of the whole work is that upon lines of swiftest descent and the isoperimetrical problems.

Maclaurin continued and developed the great work which Newton had begun, and today we are indebted to him for the discoveries and processes with which he enriched our mathematical heritage. Among his contemporaries in Scotland, Stirling alone was his equal in analytical power; and in geometry, Braikenridge alone contributed something new to his organic description of plane curves. The true significance of Maclaurin's geometrical power is seen in what it led to in the work of Poncelet, Steiner, Grassmann and Salmon.

Published Works by Maclaurin

1. Geometria Organica (dedicated in November, 1719, to Sir Isaac Newton), London, 1720.

2. Treatise of Fluxions, 2 vols., Edinburgh, 1742.

3. A Treatise of Algebra, with an Appendix De Linearum Geometricarum Proprietatibus Generalibus 1756, English translation of fifth edition, 1788. Published by Maclaurin's literary executors, Martin Folkes and Andrew Mitchell, M. P. of Abingdon, and John Hill, chaplain to Thomas Herring, Archbishop of York.

4. An account of Sir Isaac Newton's Philosophy, published by subscription by Patrick Murdoch for the benefit of Maclaurin's children, and prefaced by a memoir of Maclaurin. (First draft prepared by Maclaurin in 1728.)

5. Various communications mostly embodied in the above, and first published in the Philosophical Transactions, London, Nos. 356, 359, 394, 408, 439, 467, 469; A rule for finding the meridional parts of a spheroid, No. 461, and of the basis of the cells wherein the bees deposit their honey, No. 471.

THE TEACHING OF COLLEGE MATHEMATICS TODAY*

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1. Introduction. I believe that I should apologize for the title of my paper as it will be concerned primarily with a critique of the role the Association is playing in mathematical education. About two years ago the Board of Governors of the Association authorized its president to appoint a committee for the "coordination of studies in mathematical education." I have the honor to be a member of this committee under the highly capable and constructive chairmanship of Professor C. V. Newsom.† I must mention, and without any false modesty, that my own contribution to the progress of this committee's work has been minor indeed. I have, however, learned a few things about conditions and trends in present day mathematical education on the secondary and college levels so that my own ideas about these matters have become either modified or completely changed.

For these reasons I should make it clear that I speak for the committee only in part; most of what I have to say is an expression of my own discontent brought to an acute stage by the war and postwar conditions. This is not to say that we need look upon present day mathematical education with any great alarm; its main current flows in the right channel and the temporary fads and fashions can deflect it only temporarily. Nevertheless, we should attempt to remove all obstructions and impediments to its progress. Thus, as you know, during the war "acceleration" was all the rage and many of us "covered" the mathematical gamut from algebra to calculus in twelve weeks. As you also know, "acceleration" died a silent, but far from painless death; our chickens are coming home to roost now, and I think it is a regrettable fact, but a fact nevertheless, that most of that accelerated training was rather unsatisfactory. Now the height of style is "integration" and even if it persists, I do not believe it will leave a profound effect on mathematical education. My reason for this belief is that these integrated programs were in the main designed by humanists and educationists; mathematicians and other scientists were busy with war work at the time. I've attended a few meetings of such groups and I cannot say that their general ideas about mathematics are either very sound or very clear. It comes almost as a comic relief to read in a recent, widely accepted and publicized report of one of our foremost colleges, a comparison of the educational process with the three foci of an ellipse!

As I mentioned at the beginning it is not my purpose to tell you how to teach mathematics either on the secondary or on the college level. Neither will I presume to state what the most essential elements of any course are or, in my opinion, should be. Both of these things have recently been done in the pages of this MONTHLY, and any proposals I might make would lack the self-confidence

* Prepared for the Annual Meeting of the Association at Swarthmore, Pennsylvania, Dec. 26–27, 1946.

† Professor C. C. MacDuffee is now chairman.—EDITOR

that these papers seem to display. What I have in mind is an attempt to make mathematical instruction on the two above mentioned levels sufficiently thorough and useful to our students, and make a suggestion as to what the Association can do to improve the general mathematical atmosphere in which we live.

2. Two main trends. There are two main trends in present day mathematical education; the first is to reduce the content of a course to barest essentials. In many schools the approach is almost technological in nature. The justification for this was expressed to me by a colleague in the following terms: "If you want to teach a person how to drive a car, you do not try to teach him thermodynamics or the theory of internal combustion engines; you teach him how to push levers." And just as pushing levers will not design a better engine, so will the type of training indicated above be a very poor foundation for further study and development. It is the most difficult part of our job to make students grasp the full significance of a mathematical definition or concept; it is so much easier to learn a number of rules of procedure. I regard this type of training undesirable because the student neither obtains much useful information nor does he receive any mental stimulus or development from it.

The other trend is to relegate to lower levels instruction that traditionally was or is on a higher level. From my point of view this is a healthy trend that indicates progress in mathematical education. It is only a question of how rapidly this transition should be made, for experience and observation show that too rapid a transition is likely to prove injurious. There seems to be a desire in many quarters to start instruction in the calculus in the freshman year; such a development would relegate all preceding instruction to the high school. It seems to me that at least for the present this is not very realistic for the vast majority of high schools; also there is still the question of the maturity of the student. I don't know how desirable it would be to teach Fourier series or the Laplace Transform in the sophomore year. At any rate such reforms and progress must evidently be the joint effort of high school and college teachers of mathematics. Then, too, we receive our students from high school, and we have to build on the foundations laid down in the high school. That these foundations are often poor no one denies, but the college teacher, as a rule, does nothing to remedy these defects. We are too prone to criticize and condemn; in fact, quite recently, one of our eastern colleges held a symposium devoted to this pastime. Another group that teachers of mathematics are in the habit of damning is the group of professional educationists, the product of teachers' colleges. Quite naturally we resent the fact that there are state laws requiring teachers to "take" a certain number of educational courses in psychology and methodology; in some cases the ratio of the time spent in these courses to that spent on mathematics is 4 to 1. I know of no state law that requires a mathematics teacher to take a course in projective geometry or in number theory. It is a rare case indeed where a state board of education contains a mathematician or a person whose mathematical interest outweighs his general democratic educational interests. What

we refuse to admit is that all these groups are just as sincere in their efforts to do a good job, as we are, and are just as anxious to improve their work. In fact I am not at all sure that the college does a better job than the high school. This may sound like treason coming from a college teacher, but I know of more than one college where a student can graduate with a B.A. in mathematics without advancing any further than a first course in the calculus. I have seen their transcripts! At least the high school teachers have shown through the National Council of Teachers of Mathematics their sincere desire to improve secondary mathematical training. The Second Report of the Commission on Post War Plans which proposes "double-tracking" the high school curriculum so as to serve most effectively the various divisions of high school population is good evidence of this. And while I personally do not agree entirely either with their premises or their conclusions, nevertheless I believe that everyone of us should actively support their efforts. I am glad to say that through the efforts of Professor C. V. Newsom we are cooperating with the National Council closely and harmoniously not only in this matter but also in the matter of secondary school teacher training. Our committee is working on a program of teacher training which will improve mathematics teaching, but the adoption of the program will not come to pass unless the Association applies all the pressure it can in support of it.

3. Need for cooperation with the other fields. Another point, of the same nature, that I would like to bring to your attention is the fact that in our colleges by far the great majority of students who take courses in mathematics are primarily not students majoring in mathematics. They are usually engineers, physicists, chemists, and so forth. The mathematical part of their curriculum is almost never designed or controlled by departments of mathematics. Consider the case of E.C.P.D. (The Engineers' Council for Professional Development). They require engineering students to have had a course in solid geometry, either in high school or in college, and this quite frequently leads to a situation where a junior or a senior, after having taken a course in analytic geometry and calculus, is required to revert to his high school days. So far as I know they do not require any training in differential equations or Fourier series or any other mathematical discipline beyond a most rudimentary course in the calculus. Now while it is true that many colleges have introduced a more up-to-date mathematical program in their engineering curricula, in many cases efforts in this direction have met with failure and in some cases have led to a split so that we find in some colleges two departments of mathematics—one for engineers, the other for the rest of the college. The A.S.E.E. (The American Society for Engineering Education) has more to say about mathematical instruction for engineers than we have. The point I am trying to make is that cooperation with such an organization as E.C.P.D. or A.S.E.E. seems to offer the best chance for improvement and progress, and I believe that such cooperation must originate with the Association or it will not come about. Of course even all this will not insure more than better labels for our courses. It will not insure us against dull, poor, shallow

textbooks or dull, poor instruction. I've seen a book on high school geometry in which the only definition of a sphere I was able to find was the statement that a sphere looks like a baseball, and of course a picture of a baseball; notice the psychological approach! I have also seen a book on the calculus in which dx is defined as a small piece of x. My harsh criticism of this was parried by the rejoinder that the definition was perfectly all right since the book was written primarily for engineers. It is my belief that the Association can exert an influence for better books either by not pulling punches in its reviews or by reviewing only books that are sound. I do not mean to say that we do not have many excellent books on mathematics, but these books are not the ones that have the widest adoption.

4. Conclusion. Finally it seems to me that if the Association, in collaboration with other groups interested in mathematical education, would work out a fairly general program of study for high schools and colleges and express its disapproval of any program that is far short of it, there would be many ways of introducing such programs in various schools, or at least express its approval of those schools that are maintaining reasonable standards of training. This may sound like the establishment of an accrediting agency for mathematics but it need not be that at all. Moreover, I have seen many improvements, in other fields, that were due to the existence of such an agency. One way of insuring decent standards in mathematical education is by means of cooperative tests in achievement. As a rule administrations do not oppose that; state colleges and universities may oppose entrance examinations and, in fact, they may be illegal, but examinations in course are accepted without resistance. I do not mean that the Association should engage in the business of designing and distributing tests; there is now a Cooperative Educational Testing Commission under the chairmanship of President J. B. Conant, its aim being the study and improvement of testing procedures in all areas of learning. What I do mean is that the Association should have a clear and rather loud voice in all matters pertaining to the teaching of mathematics. It should condemn or fail to approve any outmoded or archaic practices, and it should wield all its prestige and power necessary to insure progress and improvement in mathematical education. I know of no better organization to do this, and I know there are many schools where an official stand taken by the Association would be sufficient to enable the Department of Mathematics to introduce the necessary reforms.

THE HERVEY POINT OF THE GENERAL n-LINE

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1. Introduction. An arbitrary four-line determines four triangles and if, in each of these triangles, the perpendicular bisector of the segment joining the orthocenter to the circumcenter is drawn, then a well known theorem states that the four bisectors meet in a point which has been called the Hervey point [1] of the four-line. V. Thébault, however, has pointed out [2] that S. Kantor [3] mentioned the theorem two decades before Hervey announced it. In this MONTHLY, J. R. Musselman [4] has extended the Hervey theorem to the case of n straight lines tangent to a parabola. Using the notation and methods of inversive geometry as developed by the Morleys [5], [6], we propose to obtain a generalization for an arbitrary n-line.

2. Centric center of the n-line. Given n straight lines $\Delta_1, \Delta_2, \cdots, \Delta_n$ and, in a system of complex coördinates having O as origin and Ω as base-point, let $x_1, x_2, \cdots, x_n$ be the coördinates of the images of O in these lines. Denoting the conjugate of c by $\bar{c}$, we note that $t_j = -\bar{x}_j/x_j$ is a turn. If now we consider the k lines $\Delta_1, \Delta_2, \cdots, \Delta_k$, it is easily shown that the expression

$$\begin{aligned} a_{k,i} = {} & x_1 \overset{i}{t_1}/(t_1 - t_2)(t_1 - t_3) \cdots (t_1 - t_k) \\ & + x_2 \overset{i}{t_2}/(t_2 - t_1)(t_2 - t_3) \cdots (t_2 - t_k) \\ & + \cdots, \end{aligned}$$

satisfies the relations

$$a_{k,i} = a_{k+1,i+1} - t_{k+1} a_{k+1,i} \tag{1}$$

and

$$\bar{a}_{k,i} = (-1)^k \sigma_k a_{k,k-i-1}, \tag{2}$$

where $\sigma_k = t_1 t_2 \cdots t_k$.

Since the equation of the line Δ_j is

$$\bar{x} = t_j(x - x_j),$$

the point of intersection of lines Δ_1 and Δ_2 is given by the expression $a_{2,1}$, or, using (1), we have

$$z_2 = a_{2,1} = a_{3,2} - t_2 a_{3,1}.^*$$

Hence the intersection $a_{2,1}$ lies on the circle whose parametric equation is

$$x = a_{3,2} - \tau a_{3,1},$$

τ being a turn. Since $a_{3,2}$ and $a_{3,1}$ are symmetric with respect to the three lines

* To avoid confusion, it may be noted that generally $a_{j,i}$ is not the intersection of lines Δ_j and Δ_i.

$\Delta_1, \Delta_2, \Delta_3$, it follows that this circle circumscribes the triangle formed by the three lines; its center is the point $a_{3,2}$.

Again, using (1), we may write

$$z_3 = a_{3,2} = a_{4,3} - t_4 a_{4,2},$$

and from the symmetry of the expressions $a_{4,3}$ and $a_{4,2}$ with respect to the four lines $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ we see that the circumcenters of the four triangles, formed by taking the four lines three at a time, lie on a circle, called the centric circle of the four-line. The center of the circle is the point $a_{4,3}$ and is called the centric center of the four-line.

Proceeding as before, we find that the centric centers of the five four-lines, obtained by omitting one of the five lines $\Delta_1, \Delta_2, \cdots, \Delta_5$, lie on a circle. This is the centric circle of the five-line and its center is the point $a_{5,4}$, which is called the centric center of the five-line. Continuing in this manner, we arrive at the theorem of de Longchamps [7], according to which the n centric centers of the $(n-1)$-lines belonging to an arbitrary n-line are concyclic. This circle is the centric circle of the n-line and its center

$$z_n = a_{n,n-1}$$

is the centric center of the n-line.

3. Centric focus of the n-line. The parametric equation of the centric circle of the $(n-1)$-line $\Delta_1, \Delta_2, \cdots, \Delta_{n-1}$ is

$$x = a_{n-1,n-2} - \tau a_{n-1,n-3}, \tag{3}$$

where τ is a turn. Moreover, it follows from (1) and (2) that the ratio

$$-(t_n - a_{n,2}/a_{n,1}) : (1 - t_n a_{n,n-3}/a_{n,n-2})$$

is a turn and for this value of τ the corresponding point on the circle (3) has the coördinate

$$\gamma_n = a_{n,n-1} - a_{n,2} a_{n,n-2}/a_{n,1}. \tag{4}$$

This expression being symmetric with respect to the coördinates $x_1, x_2, \cdots, x_n$, it follows that the n centric circles of the $(n-1)$-lines belonging to the given n-line pass through the point γ_n.

In the case of the four-line, the point γ_4 is easily identified as the focus of the inscribed parabola; for this reason the point γ_n is called the centric focus of the n-line. In the case of the four-line, the centric focus lies on the centric circle.

4. Orthocenter of the n-line. The orthocenter of the three-line may be thought of as the extremity of the resultant of the three vectors drawn from the circumcenter to the vertices of the triangle. Similarly, we define the orthocenter of the n-line to be the extremity of the resultant of the n vectors drawn from the centric center of the n-line to the centric centers of the $(n-1)$-lines belonging to the given n-line.

The vector having O for origin and equal to the vector joining the centric center of the n-line to the centric center of the $n-1$ lines $\Delta_1, \Delta_2, \cdots, \Delta_{n-1}$ is determined by

$$a_{n-1,n-2} - a_{n,n-1} = -t_n a_{n,n-2}.$$

Hence, it follows that the coördinate h_n of the orthocenter of the n-line is given by

$$h_n = a_{n,n-1} - s_n a_{n,n-2}, \tag{5}$$

where

$$s_n = t_1 + t_2 + \cdots + t_n.$$

5. The Hervey point of the n-line. Let us now consider the distances d and δ from the orthocenter of the n-line to the centric center and the orthocenter, respectively, of the $(n-1)$-line $\Delta_1, \Delta_2, \cdots, \Delta_{n-1}$.

The centric center of the $(n-1)$-line is given by

$$z_{n-1} = a_{n-1,n-2} = a_{n,n-1} - t_n a_{n,n-2},$$

and, using (5) and (1), we find for the orthocenter of the $(n-1)$-line

$$\begin{aligned} h_{n-1} &= a_{n-1,n-2} - s_{n-1} a_{n-1,n-3} \\ &= a_{n,n-1} - t_n a_{n,n-2} - s_{n-1}(a_{n,n-2} - t_n a_{n,n-3}) \\ &= a_{n,n-1} - s_n a_{n,n-2} + t_n s_{n-1} a_{n,n-3}. \end{aligned}$$

Combining these expressions with (5) and using (2), we get

$$\begin{aligned} d^2 &= (h_n - z_{n-1})(\bar{h}_n - \bar{z}_{n-1}) \\ &= (-1)^n \sigma_n s_{n-1} \bar{s}_{n-1} a_{n,1} a_{n,n-2}, \end{aligned}$$

and

$$\begin{aligned} \delta^2 &= (h_n - h_{n-1})(\bar{h}_n - \bar{h}_{n-1}) \\ &= (-1)^n \sigma_n s_{n-1} \bar{s}_{n-1} a_{n,2} a_{n,n-3}, \end{aligned}$$

and, therefore, their ratio is found to be

$$\lambda_n^2 = d^2/\delta^2 = a_{n,1} a_{n,n-2} / a_{n,2} a_{n,n-3},$$

which is symmetric with respect to the n coördinates $x_1, x_2, \cdots, x_n$.

Moreover, in the given n-line, the square of the radius of the centric circle is

$$a_{n,n-2} \bar{a}_{n,n-2} = (-1)^n \sigma_n a_{n,1} a_{n,n-2}, \tag{6}$$

while the square of the distance from the centric center to the centric focus is

$$a_{n,2} a_{n,n-2} \bar{a}_{n,2} \bar{a}_{n,n-2} / a_{n,1} \bar{a}_{n,1} = (-1)^n \sigma_n a_{n,2} a_{n,n-3} \tag{7}$$

and the ratio of these two squares is precisely λ_n^2. Thus we have the result:

THEOREM. *The distances from the orthocenter of an n-line to the centric center and the orthocenter of an arbitrary $(n-1)$-line, obtained by omitting one of the given lines, are in a constant ratio. This ratio equals that of the radius of the centric circle of the n-line to the length of the segment joining the centric center to the centric focus.*

In particular, when $n=4$, the centric focus lies on the centric circle and $\lambda_4=1$. Thus our result reduces to Hervey's theorem, the orthocenter of the four-line being the Hervey point.

6. Tangents to a parabola. Let the n given lines be tangent to a parabola. If the focus is taken as origin O and if the base-point Ω is the projection of O on the directrix, then the equation of the directrix is

$$x + \bar{x} = 2$$

and, since the image of O in the tangent Δ_j lies on the directrix, it follows that

$$x_j = 2/(1 - t_j).$$

Introducing the notation

$$\pi_k = (1 - t_1)(1 - t_2) \cdots (1 - t_k),$$

we can easily show that

$$z_2 = a_{2,1} = 2/\pi_2.$$

Again, since the parametric equation of the centric circle of the three-line is

$$x = a_{3,2} - \tau a_{3,1} = 2(1 - \tau)/\pi_3,$$

we find

$$a_{3,2} = a_{3,1} = 2/\pi_3.$$

In a similar way, we get

$$a_{4,3} = a_{4,2} = 2/\pi_4.$$

Also, from (1), it follows that

$$a_{3,1} = a_{4,2} - t_4 a_{4,1}$$

and, solving for $a_{4,1}$, we have

$$a_{4,3} = a_{4,2} = a_{4,1} = 2/\pi_4.$$

Continuing in this manner, we finally obtain

$$a_{n,n-1} = a_{n,n-2} = \cdots = a_{n,1} = 2/\pi_n.$$

From (4) it now follows that $\gamma_n=0$, which means that the centric focus of the n-line is the focus of the parabola; moreover, it is on the centric circle and, therefore, $\lambda_n=1$. Thus we have the theorem:

THEOREM. *If n straight lines are tangent to a parabola, then the segments joining the centric centers to the orthocenters of the $(n-1)$-lines, formed by omitting one of the lines, are such that the perpendiculars erected at their mid-points meet in a point* [4].

This point is the orthocenter of the n-line and, from (5), its coördinate is

$$h_n = 2(1 - s_n)/\pi_n.$$

7. Tangents to a deltoid. Given n lines tangent to a deltoid. If the in-circle is taken as base-circle and if Ω is the symmetric image with respect to O of an apse of the curve, then the line Δ_j is given by the equations

$$x - \bar{x}/t_j = t_j - 1/t_j^2$$

and

$$x_j = t_j - 1/t_j^2,$$

where t_j is the mid-point of the segment cut off on the line by the deltoid.

By direct computation and using (1), we find

$$a_{2,1} = t_1 + t_2 + 1/t_1t_2 = s_2 + 1/\sigma_2,$$

$$a_{3,2} = t_1 + t_2 + t_3 = s_3, \qquad a_{3,1} = 1 - 1/t_1t_2t_3 = 1 - 1/\sigma_3,$$

$$a_{4,3} = s_4, \qquad a_{4,2} = 1, \qquad a_{4,1} = 1/\sigma_4.$$

Continuing in this manner, we are able to prove that, when $n \geqq 5$,

$$a_{n,n-1} = s_n, \qquad a_{n,n-2} = 1, \qquad a_{n,n-3} = 0.$$

These results combined with (5) lead to the theorem:

THEOREM. *Four or more tangents to a deltoid form an n-line, whose orthocenter coincides with the center O of the deltoid.*

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MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.

SOME REMARKS ON EULER'S TOTIENT

V. L. KLEE, JR., University of Virginia

Ratat [1] and Goormaghtigh [2] have recorded values of n for which $\phi(n)=\phi(n+1)$. A list of all such values of $n<3000$, found from Glaisher's table [3] follows.

$\phi(n)=\phi(n+1)$ for $n=1$, 3, 15, 104, 164, 194, 255, 495, 584, 975, 2204, 2625, and 2834.

The last five entries are new. It is noteworthy that for all the stated values except 1 and 3, whichever of n and $n+1$ is odd is divisible by 15.

Ratat [1] also noted that for $n<125$, $\phi(2n\pm1)\geqq\phi(2n)$. We list here all values of $n<1500$ for which this inequality fails to hold:

$\phi(2n-1)<\phi(2n)$ for $n=263$, 293, 368, 578, 683, 743, 788, 878, 893, 908, 998, 1073, 1103, 1208, 1238, 1268, 1403, and 1418.

$\phi(2n+1)<\phi(2n)$ for $n=157$, 262, 367, 412, 577, 682, 787, 877, 892, 907, 997, 1072, 1207, 1237, 1312, and 1402.

Now if p is an odd prime and $2p-1$ is prime, then $\phi(n)=\phi(n+2)$ for $n=2(2p-1)$. There are also the following values of $n<3000$ for which $\phi(n)=\phi(n+2)$ although n and $n+2$ are not related in the manner just indicated: 4, 7, 8, 32, 70, 308, 512, 572, 635, 728, 2170, and 2695.

Finally, we note that if $n<3000$ and $\phi(n)+2=\phi(n+2)$, then either n and $n+2$ are prime or n has the form $4p$, where p and $2p+1$ are prime.

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GAUSSIAN MEANS

WILLIAM GUSTIN, University of California, Los Angeles

Let

$$A = (a_1, \cdots, a_n)$$

be a sequence of n positive real numbers, and let

$$Q = (q_1, \cdots, q_n)$$

be another sequence of n positive real numbers, called weights, such that

$$\sum_{i=1}^{n} q_i = 1.$$

For any real number t the mean of order t of the sequence A weighted by the sequence Q is defined as

$$\mathfrak{M}(t, A, Q) = \phi_t^{-1}\left[\sum_{i=1}^{n} q_i \phi_t(a_i)\right],$$

where the function ϕ_t with inverse ϕ_t^{-1} is given by

$$\phi_t(a) = \begin{cases} \log a, & t = 0, \\ a^t, & t \neq 0, \end{cases}$$

for every positive real argument a. It is known that $\mathfrak{M}(t, A, Q)$ is a nondecreasing continuous function of t whose g.l.b. is min A and whose l.u.b. is max A [1].

We now generalize this mean of order t to a mean of order T, where

$$T = (t_1, \cdots, t_n)$$

is any sequence of n real numbers. Let the sequences

$$A^k = (a_1^k, \cdots, a_n^k), \qquad (k = 0, 1, 2, \cdots),$$

be defined recursively as follows. Put

$$A^0 = A.$$

Then determine the elements of the sequence A^{k+1} from the sequence A^k by the formula

$$a_i^{k+1} = \mathfrak{M}(t_i, A^k, Q), \qquad (i = 1, \cdots, n;\ k \geqq 0).$$

We shall prove that for a fixed index i the infinite sequence

$$A_i = (a_i^1, a_i^2, \cdots)$$

has a limit independent of i. We call this common limit the mean of order T of the sequence A weighted by Q, and denote it by

$$\mathfrak{M}(T, A, Q).$$

It is our purpose here merely to establish the existence of this mean and not to investigate its properties.

The particular mean

$$\mathfrak{M}[(0, 1), (a_1, a_2), (\tfrac{1}{2}, \tfrac{1}{2})]$$

is known as the arithmetico-geometric mean of a_1 and a_2; it was first investigated by Gauss in connection with a problem in potential theory [2]. The special methods which have been employed to prove the convergence of the infinite sequences A_1 and A_2 to the arithmetico-geometric mean of a_1 and a_2 do not carry over to the more general mean discussed here.

For convenience we permute the indices of each of the sequences T, A, Q

so that T is nondecreasing. Thus, in particular,

$$t_1 \leqq t_i \leqq t_n.$$

Since the mean of order t is a nondecreasing function of t, we see that

$$a_1^k \leqq a_i^k \leqq a_n^k, \qquad (k \geqq 1). \tag{1}$$

Therefore, the sequence A^k is bounded below by a_1^k and is bounded above by a_n^k, so that

$$a_1^k \leqq \mathfrak{M}(t_1, A^k, Q) = a_1^{k+1} \leqq a_n^{k+1} = \mathfrak{M}(t_n, A^k, Q) \leqq a_n^k, \qquad (k \geqq 1).$$

The bounded nondecreasing infinite sequence A_1 then has a limit α_1, and similarly the bounded nonincreasing infinite sequence A_n has a limit α_n. Taking inferior and superior limits of (1), we have

$$\alpha_1 = \underline{\lim}_{k\to\infty} a_1^k \leqq \underline{\lim}_{k\to\infty} a_i^k \leqq \overline{\lim}_{k\to\infty} a_i^k \leqq \overline{\lim}_{k\to\infty} a_n^k = \alpha_n. \tag{2}$$

From (2) and the fact that the function $\phi_{t_1}(a)$, written below as $\phi(a)$, is a continuous strictly increasing function of a provided $t_1 \geqq 0$, we obtain the following extended inequality:

$$\begin{aligned}
\phi(\alpha_1) &= \phi(\underline{\lim}_{k\to\infty} a_1^{k+1}) \\
&= \underline{\lim}_{k\to\infty} \phi(a_1^{k+1}) \\
&= \underline{\lim}_{k\to\infty} \sum_{i=1}^{n} q_i \phi(a_i^k) \\
&\geqq \sum_{i=1}^{n} q_i \phi(\underline{\lim}_{k\to\infty} a_i^k) \\
&\geqq \sum_{i=1}^{n-1} q_i \phi(\alpha_1) + q_n \phi(\alpha_n) \\
&= \sum_{i=1}^{n} q_i \phi(\alpha_1) + q_n [\phi(\alpha_n) - \phi(\alpha_1)] \\
&= \phi(\alpha_1) + q_n [\phi(\alpha_n) - \phi(\alpha_1)] \\
&\geqq \phi(\alpha_1).
\end{aligned}$$

Subtracting $\phi(\alpha_1)$ from this inequality we get

$$0 \geqq q_n [\phi(\alpha_n) - \phi(\alpha_1)] \geqq 0,$$

whence $\alpha_1 = \alpha_n$, if $t_1 \geqq 0$. If $t_1 < 0$, the function $\phi(a)$ is a continuous strictly decreasing function of a, and a similar argument, in which we use superior limits and reverse the inequality signs, also shows that $\alpha_1 = \alpha_n$. Finally we see from (2) that this common limit $\alpha = \alpha_1 = \alpha_n$ is the limit of each of the infinite sequences

A_i, so that

$$\alpha = \mathfrak{M}(T, A, Q).$$

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CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania. Contributions are invited on topics of immediate interest to teachers of undergraduate mathematics.

OBJECTIVES IN CALCULUS

C. C. MACDUFFEE, University of Wisconsin

What should be the objectives in a beginning course in the calculus? That is a question which many college teachers ask themselves, and to which it is difficult to frame an answer. Calculus is the course for which the student has long been preparing through college algebra, trigonometry and analytics, and for many a student it is the last mathematics course which he will ever take. The amount of interesting and valuable material which is at that point open to him is large and beyond the capacity of the time available for its complete presentation. What gems shall be presented and which omitted is a problem which we all have to face.

Our problem is complicated by the fact that no two students have exactly the same backgrounds, interests, personalities or plans for the future. Education should be a very personal matter between Professor A and Student X. The ideal college would give to each student a tailor-made course fitted to his exact needs and capacities, allowing him to proceed as rapidly and deeply as his abilities allow. But such methods under competent teachers are too expensive for any modern college so that the student must be fitted as best he may with a ready-made suit of clothes. The problem is to design the suit so that it will fit the largest number of students reasonably well.

The first objective in a course in calculus has to be the basic techniques of differentiation and integration. Just as the fundamentals of spelling and grammar have to be learned before one can compose literature, so these techniques have to be acquired before one can use the calculus. Any student who can qualify to enter a class in calculus can with patience master these techniques. The pity of it is that so many students get not much else for their labors.

Beyond the fundamental techniques, there seems to be a difference of opinion as to the best procedure. There seems to be one school of thought which would have the student solve large numbers of problems in mechanics and physics without much attempt at incisive reasoning or rigor, apparently with the idea that after enough experience the student's subconscious mind will take over and set up the problem for him. This is known as "standard engineering practice."

There is another school which tends to minimize the applications and to teach the calculus as a pure and lofty discipline of the mind. Most of the class time is spent on real variable theory, and a few students may even learn to throw ϵ's and δ's around with what the teacher believes (until the day of the final examination) to be a fair degree of intelligence. This method is in great favor with young Ph.D.'s. It is not in favor among deans.

The ideal method obviously lies in neither of these directions. The mathematician cannot afford to forget that the calculus was developed for the purpose of solving problems in mechanics and physics, and that its greatest glory even now is in connection with the applications. The engineer too must remember that Newton and Laplace and the Bernoullis were deep and incisive thinkers whose intuitions were merely the manifestations of careful and rigorous thinking.

If we grant that the education of a scientist consists in the development of the power to do, we must admit that our proper goal in the calculus is to develop the student's ability to interpret the physical world in mathematical terminology. This presupposes of course that he shall be able to speak the language of mathematics, that he shall have a command of the techniques and also of the theory behind these techniques. But over and beyond this fluency, he must have achieved an intuitive feeling for the elementary concepts of mechanics and physics, and his thinking along these lines must be in the language of mathematics. This is a large order for a first course in calculus, and obviously incapable of complete achievement. But I believe it is a measuring rod upon which the success of such a course must be judged.

There is a pedagogical sequence for the presentation of ideas which seems to be inherent in the human animal, and which seems to be quite unrelated to what we consider the logical order. Successful teaching respects this pedagogical order. Thus the modern foundations of the calculus, which make the concept of limit a purely static concept without appeal to the notions of time or motion, is magnificent. Every graduate student should be required to master it. But it should not be the calculus teacher's one and only god.

One of the weaknesses of American universities is their intense departmentalization. Nature does not recognize the fine distinctions between what belongs to mathematics, what to physics, and what to chemistry. Neither did the great universities of Europe to the extent that we do. We sometimes teach calculus with no applications to physics, we frequently try to teach physics without using mathematics. What is perhaps even more demoralizing to the students, our mathematics teachers sometimes demonstrate an incompetence in physics and vice versa.

There have been attempts to coördinate the teaching of calculus and mechanics to the extent of preparing textbooks in the two courses which can be used in the two courses simultaneously, the calculus being available by the time it is needed in mechanics, and the problems in mechanics ready to illustrate the theorems of the calculus. Perhaps by the time our elective system has been modified so that the average student does not have an irregular program, this scheme will be more widely tried.

In these days of educational experimentation, various combinations of courses are being tried which have never been tried before. What could be more natural than a combination course of basic physics and calculus? This course would probably have to be spread over two years if it were to contain a complete course in both physics and calculus. It would have to be given by a man who is competent in and sympathetic toward both courses. He could not be a physicist who teaches a little mathematics as a "tool," nor a mathematician who "runs in a few illustrations from physics." But he would be able to develop physicists to whom mathematics is a mother tongue. Can you think of a better background for scientists of the present age?

Regardless of the framework in which it is taught, the first course in calculus must be handled with a fine sense of balance. It should be rigorous up to the capacity of the student to appreciate rigor, and this rigorous treatment must be extended to the problems, not merely confined to the proof of the existence of the definite integral. But the fundamental and basic problem is to develop the student's intuitions so that mathematics is to him a spoken language. Then and only then is he in a position to appreciate the meaning of rigor. For is rigor anything else than clarity?

FORMULA FOR THE AREA OF A TRIANGLE

M. K. Fort, Jr., University of Virginia

We prove in Theorem 1 that a certain determinant is an invariant. Theorem 1 is then used to prove the well known formula

$$A = 1/2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

for the area of a triangle.

THEOREM 1. *If P_1, P_2, P_3 are points in a plane, and these points have coördinates (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_1', y_1'), (x_2', y_2'), (x_3', y_3') respectively in rectangular coördinate systems C and C' (which we shall assume to have the orientation commonly used in analytic geometry texts); then*

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix}.$$

The transformation from the C' system to the C system is given by equations of the form

$$x = mx' - ny' + h$$
$$y = nx' + my' + k$$

where $m^2+n^2=1$. Therefore

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} mx_1' - ny_1' + h & nx_1' + my_1' + k & 1 \\ mx_2' - ny_2' + h & nx_2' + my_2' + k & 1 \\ mx_3' - ny_3' + h & nx_3' + my_3' + k & 1 \end{vmatrix}.$$

The determinant on the right side of the above equation can be simplified by subtracting the proper multiples of the last column from the first two columns. If we do this we get

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} mx_1' - ny_1' & nx_1' + my_1' & 1 \\ mx_2' - ny_2' & nx_2' + my_2' & 1 \\ mx_3' - ny_3' & nx_3' + my_3' & 1 \end{vmatrix}.$$

The determinant on the right is equal to

$$(m^2 + n^2) \begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix}.$$

Since $m^2+n^2=1$, we see that

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{vmatrix}.$$

THEOREM 2. *If P_1, P_2, P_3 are the vertices of a triangle and the cyclic order $P_1P_2P_3P_1$ induces a counter-clockwise orientation on the boundary of the triangle, then the area A of the triangle is given by*

$$A = 1/2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Choose a coördinate system C' so that P_1 is at the origin and P_2 is on the positive x'-axis. It follows from the fact that $P_1P_2P_3P_1$ induces a counter-clockwise orientation, that P_3 must be in either the first or second quadrant. For this choice of C' we now see that $x_1' = y_1' = y_2' = 0$, that x_2' is the length of the side P_1P_2, and that y_3' is the length of the altitude perpendicular to this side. Thus the area

of the triangle satisfies $2A = x_2' y_3'$. We now apply Theorem 1 and obtain

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ x_2' & 0 & 1 \\ x_3' & y_3' & 1 \end{vmatrix} = x_2' y_3' = 2A.$$

Therefore

$$A = 1/2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

In a similar fashion we can prove:

THEOREM 3. *If P_1, P_2, P_3 are the vertices of a triangle and the cyclic order $P_1P_2P_3P_1$ induces a clockwise orientation on the boundary of the triangle, then the area A of the triangle is given by*

$$A = -1/2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 776. *Proposed by L. R. Ford, Illinois Institute of Technology*

"Are those your children that I hear playing in the garden?" asked the visitor.

"There are really four families of children," replied the host. "Mine is the largest, my brother's family is smaller, my sister's is smaller still, and my cousin's is the smallest of all. They are playing drop the handkerchief," he went on; "they prefer baseball but there are not enough children to make two teams. Curiously enough," he mused, "the product of the numbers in the four groups is my house number, which you saw when you came in."

"I am something of a mathematician," said the visitor, "let me see whether I can find the numbers of children in the various families." After figuring for a

time he said, "I need more information. Does your cousin's family consist of a single child?" The host answered his question, whereupon the visitor said, "Knowing your house number and knowing the answer to my question, I can now deduce the exact number of children in each family."

How many children were there in each of the four families?

E 777. *Proposed by C. R. Perisho, McCook Junior College*

Find the number of permutations of n objects with the restriction that in no arrangement may an object be adjacent to either of its neighbors in the original order.

E 778. *Proposed by Victor Thébault, Tennie, Sarthe, France*

For a given tetrahedron $ABCD$, find the point P in space such that the shortest paths separating P from each of the vertices A, B, C, D, after having touched the opposite faces BCD, CDA, DAB, ABC, are equal to each other.

E 779. *Proposed by P. A. Pizá, San Juan, Puerto Rico*

Let x, y, z be three positive integers and set

$$a = x + y, \qquad b = x + z, \qquad c = x + y + z.$$

Show that for any prime exponent $p > 2$,

$$(ab)^p - (cx)^p - (yz)^p$$

is divisible by the product $pabcxyz$.

E 780. *Proposed by G. Pólya, Stanford University*

A lampshade has the shape of a frustum of a right circular cone. Its perimeter is P at the bottom, p at the top, and its slant height is s. Show that such a lampshade can be cut out in one piece from a rectangular sheet of paper with dimensions

$$P \quad \text{and} \quad s + p(P - p)/8s.$$

You can even save paper for a flap to glue the two ends together, except in the limiting case where $P = p$, when not a bit of paper is wasted.

SOLUTIONS

Bounds for Finite Harmonic Series

E 746 [1946, 591]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Show that for all positive integers r

$$r\{1 - (n+1)^{-1/r}\} < 1/1 + 1/2 + \cdots + 1/n \leqq r(n^{1/r} - 1) + 1.$$

I. *Solution by J. H. Simester, University of Louisville.* We shall use the function

$$f(n) = 1 + 1/2 + \cdots + 1/n - \ln(1 + n),$$

(whose limit when $n\to\infty$ is Euler's constant). A property of $f(n)$ is that

$$0 < f(n) < 1 - 1/(1+n),$$

from which it follows that, for $n>1$,

$$f(n-1) + 1/n < 1 - 1/n + 1/n = 1.$$

Now, from

$$\int_1^{1+n} t^{-1-1/r}dt < \int_1^{1+n} t^{-1}dt = \ln(1+n),$$

it follows that

$$r\{1-(1+n)^{-1/r}\} < 1 + 1/2 + \cdots + 1/n \tag{1}$$

for any positive integral values of r and n. Also, for $n>1$,

$$1 + 1/2 + \cdots + 1/n = \ln n + \{f(n-1) + 1/n\} < \ln n + 1$$
$$= \int_1^n t^{-1}dt + 1 < \int_1^n t^{-1+1/r}dt + 1 = r(n^{1/r} - 1) + 1,$$

whence, for all positive integers r and n

$$1 + 1/2 + \cdots + 1/n \leqq r(n^{1/r} - 1) + 1. \tag{2}$$

Results (1) and (2) are the desired inequalities.

II. *Solution by the Proposer.* The identity

$$\frac{(m+1) - m}{(m+1)^{1/r} - m^{1/r}} = \sum_{i=1}^{r} (m+1)^{1-i/r} m^{(i-1)/r}$$

gives

$$\frac{(m+1) - m}{(m+1)^{1/r} - m^{1/r}} \leqq r(m+1)^{1-1/r} \leqq r(m+1), \qquad m \geqq 0.$$

Similarly

$$\frac{\dfrac{1}{m} - \dfrac{1}{m+1}}{\left(\dfrac{1}{m}\right)^{1/r} - \left(\dfrac{1}{m+1}\right)^{1/r}} = \sum_{i=1}^{r} \left(\frac{1}{m}\right)^{1-i/r} \left(\frac{1}{m+1}\right)^{(i-1)/r}$$

gives

$$\frac{\dfrac{1}{m} - \dfrac{1}{m+1}}{\left(\dfrac{1}{m}\right)^{1/r} - \left(\dfrac{1}{m+1}\right)^{1/r}} > \frac{r}{(m+1)^{1-1/r}} > \frac{r}{m+1}, \qquad m > 0.$$

Hence, for $m \geqq 0$,

$$r\{(m+1)^{-1/r} - (m+2)^{-1/r}\} < 1/(m+1) \leqq r\{(m+1)^{1/r} - m^{1/r}\}.$$

Summation with respect to m now leads to the required result.

III. *Solution by Barney Bissinger, Fitchburg, Massachusetts.* The left hand member of the inequalities is equal to the area under the curve $y=(x+1)^{-1-1/r}$, above the x-axis, and between the ordinates at $x=1$ and $x=n$. This area is obviously less than $\sum_1^n 1/i$.

Similarly, the right hand member of the inequalities is equal to 1 plus the area under the curve $y=x^{-1+1/r}$, above the x-axis, and between the ordinates at $x=1$ and $x=n$. This area is clearly greater than $1+\sum_2^n 1/i$; equality occurs only when $r=n=1$.

Also solved by Paul Brock, Ragnar Dybvik, and Norman Miller.

Diophantine Vectors

E 747 [1946, 591]. *Proposed by H. W. Becker, Omaha, Nebraska*

Can we decompose an integer force F into n integer forces $F_1, \cdots, F_n$, such that the sum of any number of the components is also an integer force, where $n>2$?

Editorial Note. The answer is *yes*. For consider a tetrahedron $ABCD$ in which $AC=AD=BA=BC=BD=3$, and $CD=4$. Consider the three vectors $F_1 \equiv \overrightarrow{AB}$, $F_2 \equiv \overrightarrow{BC}$, $F_3 \equiv \overrightarrow{CD}$. Then it is easily seen that $F_1+F_2+F_3$, F_2+F_3, F_3+F_1, F_1+F_2 all have integral magnitudes. Other solutions are of course possible.

Becker has also proposed the problem of determining the existence or nonexistence of $n>2$ mutually orthogonal integer vectors such that the sum of any number of them will also be an integer vector. This problem seems much more difficult and possesses a considerable associated literature.

Lateral Area of an Oblique Cone of Revolution

E 748 [1946, 591]. *Proposed by George Pólya, Stanford University*

Let α be the angle between the axis and any element of a right circular cone. If the cone is cut through by a plane (not necessarily perpendicular to the axis) show that the lateral area of the part remaining between the plane and the vertex is given by $\pi AG \sin \alpha$, where A and G are respectively the arithmetic and the geometric mean of the longest and the shortest remaining elements.

Solution by G. K. Klausner, Cooper Union Institute of Technology. The required area S is easily shown to be $E \csc \alpha$, where E is the projection of S on a plane perpendicular to the axis of the cone. From geometry the semi-axes of the ellipse E are found to be $\frac{1}{2}(a+b) \sin \alpha$ and $\sqrt{ab} \sin \alpha$, where a and b are the lengths of the longest and shortest elements. Since $A=\frac{1}{2}(a+b)$ and $G=\sqrt{ab}$, $E=\pi AG \sin^2 \alpha$ and $S=\pi AG \sin \alpha$.

Also solved by F. Ballantine, D. H. Browne, H. E. Fettis, Free Jamison, D. W. Matlock, R. K. Morley, D. K. Pease, C. F. Pinzka, P. W. A. Raine, J. H. Simester, A. Sisk, P. D. Thomas, R. H. Urbano, and the proposer. Thomas noted that this problem occurs as example 13, p. 380 of Bowser's *Differential and Integral Calculus* (1910).

A Cone Inscribed in a Sphere

E 749 [1946, 591]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a given sphere inscribe a right circular cone whose lateral area is equal to the area of the zone beneath its base. Show that the total area of the cone is equal to the area of the zone in which it is inscribed.

Solution by the Proposer. Let S be the vertex of the cone, $SH=x$, its altitude, piercing the sphere again at D, and SA any arbitrary element of the cone. Let d be the diameter of the sphere. The lateral area of the cone is given by

$$\pi(HA)(SA) = \pi\sqrt{x(d-x)dx}$$

and that of the zone beneath the cone by

$$\pi(DA)^2 = \pi d(d-x).$$

The relation $\pi(HA)(SA)=\pi(DA)^2$ gives the equation

$$x\sqrt{d(d-x)} = d(d-x),$$

which reduces to

$$x^2 - d(d-x) = 0.$$

Without the need of solving, this last equation shows that *the point H divides the diameter SD of the sphere in mean and extreme ratio.* There is thus a classical geometrical construction of the point H, and consequently a construction of the required cone.

The second part of the problem follows from the relations

$$\pi(HA)(SA) + \pi(HA)^2 = \pi dx = \pi(SA)^2.$$

The solution of this problem implicitly contains the solutions of the following problems:

1. To inscribe in a given sphere a right circular cone such that the lateral area and the total area have a sum equal to the area of the sphere.

2. To cut a sphere by a plane such that the area of the section will be equal to the difference of the areas of the zones determined by the plane.

3. To cut a sphere by two parallel planes such that the area of each section will be equal to that of the zone intercepted by the planes.

4. The pole P, with respect to the sphere, of the base of the cone is such that $PH=2d$, and the distance $PD=d(\sqrt{5}+1)/2$ is equal to twice the side of a regu-

lar 3/10 star inscribed in a great circle of the given sphere.

Also solved by W. E. Buker, G. Y. Cherlin, Ragnar Dybvik, D. H. Erkiletian, Jr., R. B. Herrera, B. R. Leeds, D. K. Pease, C. F. Pinzka, P. W. A. Raine, C. D. Smith, and P. D. Thomas.

Remarks by C. D. Smith, Mississippi State College. The problem of finding the inscribed right circular cone with a volume equal to that of the spherical segment below leads to the cubic

$$2h^3 + Rh^2 - 4R^2h + R^3 = 0,$$

where R is the radius of the sphere and h is the distance from the center of the sphere to the base of the cone. Removing the undesired root $h=R$, we find $h=R(\sqrt{17}-3)/4$. The similar distance h for the given problem is $R(\sqrt{5}-2)$. Using these values of h to calculate the radii of the bases of the cones for the two problems we find that these radii differ from each other by only (approximately) $0.012R$.

More surprising is the case of a spheroid with axis of rotation $2R$. If we take the inscribed cone with altitude $R+h$ along the axis $2R$, and insist that the volume of the cone be equal to that of the segment beneath the base, we find the same cubic as for the sphere. This equation is independent of the eccentricity of the generating ellipse, and hence we have cones of equal altitudes for all spheroids having axis of rotation equal to $2R$.

Editorial Note. It might be apropos to mention here a curiosity sent some time ago to this department by T. A. Bickerstaff of the University of Mississippi: *The vertex angle of the maximum inscribed cone in a sphere equals a base angle of the minimum circumscribed cone about the sphere.* The proof is very straightforward.

Interior Diagonal Points

E 750 [1946, 591]. *Proposed by Paul Erdös, Syracuse University*

Find the number of intersections of the diagonals of a convex polygon of n sides.

I. *Solution by Norbert Kaufman and R. H. Koch, Chicago, Illinois.* Consider a convex polygon of $n \geqq 4$ sides. Every combination of the n vertices taken four at a time determines a quadrilateral which has two intersecting diagonals. Also, every two intersecting diagonals of the polygon determine a quadrilateral. Therefore the required number of intersections is $\binom{n}{4}$. These intersections, of course, may not all be distinct.

II. *Solution by Arthur Rosenthal, University of New Mexico.* If k vertices lie on one side of a diagonal d, then $n-k-2$ vertices lie on the other side, and hence d intersects $k(n-k-2)$ other diagonals. Taking into account that each diagonal is determined by two vertices and each intersection by two diagonals, one finds

that the number of intersections of the diagonals of a simple convex polygon of n sides is

$$(n/4)\sum_{k=1}^{n-3} k(n-k-2) = \binom{n}{4},$$

not counting the n vertices of the polygon.

A much more difficult problem is to find the number of distinct points of intersection of the diagonals of a simple regular polygon of n sides.

Also solved by D. W. Alling, Fred Ballantyne, H. W. Becker, D. H. Browne, F. A. Butter, Jr., G. Y. Cherlin, Monte Dernham (two ways), Harley Flanders, William Gustin, J. B. Kelly, D. W. Matlock, W. E. Patten, C. F. Pinzka, and the proposer.

Several solvers also found the number of exterior diagonal points. Dernham pointed out the interesting fact that, as n increases, the decreasing ratio of interior to exterior intersections,

$$(n-1)(n-2)/2(n-4)(n-5),$$

approaches the limit 1/2. He also suggested the problem of finding the *minimum* possible number of distinct interior diagonal points that may be possessed by a convex n-gon. Patten pointed out that the proposed problem occurs as ex. 7, p. 34, vol. II (2nd ed.) of Chrystal's *Text Book of Algebra*. As an allied problem he suggested proving that all interior diagonal points of a regular convex odd-sided polygon are simple intersections. As a still more difficult problem he proposed that of determining the existence or non-existence of n-gons possessing prescribed numbers of diagonal points of given multiplicity. If I_j designates the number of interior diagonal points of multiplicity j, then $(I_2, I_3, \cdots, I_m)$ may be called the *index* of the n-gon. For $n=7$ Patten stated that $I_m=0$ for $m>3$, and that 7-gons exist for $(I_2, I_3)=(35, 0), (32, 1), (29, 2)$, and $(26, 3)$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4254. *Proposed by Paul Erdös, Syracuse University*

We have seven points in the plane. Prove that we can always select three

which do not form an isosceles triangle. For six points this does not necessarily hold. (If A, B, C are on a line we can define that they do not form an isosceles triangle if $AB \neq BC$.)

4255. *Proposed by G. Pólya, Stanford University*

A sequence $\{x_n\}$ is defined recursively, in terms of two numbers x_0 and x_1, by the formula

$$x_n = \frac{(n-1)g}{1+(n-1)g}\,x_{n-1} + \frac{1}{1+(n-1)g}\,x_{n-2},$$

where g is a given positive quantity. Find an expression for the limit of x_n as $n \to \infty$. (This generalizes problem E 694 (1945, 516) which corresponds to the special case $g=1$.)

4256. *Proposed by N. A. Court, University of Oklahoma*

Given a sphere orthogonal to two circles lying in two distinct planes. If the center of the sphere is conjugate, with respect to one of the circles, to the point in which the plane of that circle cuts the axis of the other circle, the same is true of the center of the sphere, if the roles of the two circles are interchanged.

Note. A circle is orthogonal to a sphere if the plane of the circle cuts the sphere along a great circle orthogonal to the given circle (see, for instance, the Proposer's *Modern Pure Solid Geometry*, p. 138, art. 416).

4257. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In any arithmetic progression show that the difference (sum) of the product of n consecutive terms and the product of n other consecutive terms is always divisible, provided n is even (odd), by the sum of the greatest and least of the terms. Prove also the corollary: If a and b are positive integers and if $a+b+1=p$ is prime, then

$$a!b! \pm 1 \equiv 0 \pmod{a+b+1},$$

the sign being $+$ or $-$ according as a and b are even or odd.

4258. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that the necessary and sufficient condition that four non-collinear points are such that each is the orthocenter of the other three, is

$$\pm 34\cdot 42\cdot 23 \pm 41\cdot 13\cdot 34 \pm 12\cdot 24\cdot 41 \pm 23\cdot 31\cdot 12 = 0,$$

where rs denotes the distance between the rth and sth points, and three of the signs differ from the fourth.

SOLUTIONS

NOTE. 4183 (1947–235) was proposed by P. M. Hummel, University of Alabama, rather than Cezar Coşniţa as stated in error.

Binomial Coefficients

4189 [1946, 103]. *Proposed by Albert Wilansky, Brown University*

Prove that

$$\sum_{r=0}^{m}(-1)^r\binom{m+k+\alpha}{r+k+\alpha}\binom{r+k}{k}=\binom{m+\alpha-1}{m}, \qquad \alpha \geqq 0.$$

Solution by S. F. Lee, Yenching University, Peiping, China. Recalling the expansion of $(1-x)^{\mu}$, we have

$$(1-x)^{m+k+\alpha}=\sum_{n=0}^{\infty}(-1)^n\binom{m+k+\alpha}{n}x^n=\sum_{n=0}^{\infty}(-1)^n\binom{m+k+\alpha}{m-n+k+\alpha}x^n,$$

$$(1-x)^{-k-1}=\sum_{n=0}^{\infty}(-1)^n\binom{-k-1}{n}x^n=\sum_{n=0}^{\infty}\binom{k+n}{k}x^n,$$

and their product

$$(1-x)^{m+\alpha-1}=\sum_{n=0}^{\infty}(-1)^n\binom{m+\alpha-1}{n}x^n.$$

Equating the coefficients of x^m, we obtain

$$\sum_{r=0}^{m}(-1)^{m-r}\binom{m+k+\alpha}{r+k+\alpha}\binom{r+k}{k}=(-1)^m\binom{m+\alpha-1}{m},$$

which is equivalent to the proposed result. We note that the restriction $\alpha \geqq 0$ has not been used in the proof.

Solved also by D. W. Alling, W. J. Combellack, H. S. Grant, J. R. Kinney, M. S. Knebelman, H. L. Krall, A. S. Peters, James Singer, N. Wyman, and the Proposer.

Summation

4191 [1946, 103]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$\frac{1}{\cosh(\pi/2)}-\frac{1}{3\cosh(3\pi/2)}+\frac{1}{5\cosh(5\pi/2)}-\cdots=\frac{\pi}{8}.$$

Solution by H. E. Fettis, Dayton, Ohio. The Fourier series for $\sinh(bx)$, obtained by the standard procedure, is

$$\sinh(bx)=\frac{2\sinh(b\pi)}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n\sin(nx)}{b^2+n^2}.$$

Therefore

$$\frac{\pi}{2}\,\frac{\sinh(bx)}{\sinh(b\pi)} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n\sin(nx)}{b^2+n^2}.$$

Since the series is valid for $-\pi \leqq x \leqq \pi$, we may replace x by $\pi - x$, and obtain

$$\frac{\pi}{2}\,\frac{\sinh(b\pi - bx)}{\sinh(b\pi)} = \sum_{n=1}^{\infty}\frac{n\sin(nx)}{b^2+n^2}.$$

Adding the two series, and noting that all terms with n even cancel, we obtain the result

$$\frac{\pi}{4}\,\frac{\cosh\frac{1}{2}b(\pi-2x)}{\cosh\frac{1}{2}b\pi} = \sum_{n=0}^{\infty}\frac{(2n+1)\sin(2n+1)x}{b^2+(2n+1)^2},$$

and, in particular if $x=\pi/2$

$$\operatorname{sech}\tfrac{1}{2}b\pi = \frac{4}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{2n+1}{b^2+(2n+1)^2}.$$

The series under consideration may therefore be written

$$\begin{aligned}
&\sum_{m=0}^{\infty}(-1)^m\frac{\operatorname{sech}\frac{1}{2}(2m+1)\pi}{(2m+1)} \\
&\qquad = \frac{4}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}(2n+1)}{(2m+1)\left[(2n+1)^2+(2m+1)^2\right]} \\
&\qquad = \frac{4}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+n}}{2n+1}\left[\frac{1}{2m+1}-\frac{2m+1}{(2n+1)^2+(2m+1)^2}\right] \\
&\qquad = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left[\frac{\pi}{4}-\frac{\pi}{4}\operatorname{sech}\tfrac{1}{2}(2n+1)\pi\right] \\
&\qquad = \sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{2n+1}-\frac{1}{2n+1}\operatorname{sech}\tfrac{1}{2}(2n+1)\pi\right] \\
&\qquad = \frac{\pi}{4}-\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}\operatorname{sech}\tfrac{1}{2}(2n+1)\pi.
\end{aligned}$$

Since the two summations are identical, the desired result follows immediately.

Note by C. D. Olds, San Jose College, California. The stated result follows immediately from a theorem given by Ramanujan in his famous letter to G. H. Hardy in 1913, namely:

$$\int_0^{\infty}\frac{\sin 2nx}{x(\cosh\pi x+\cos\pi x)}\,dx = \frac{\pi}{4}-2\left(\frac{e^{-n}\cos n}{\cosh\pi/2}-\frac{e^{-3n}\cos 3n}{3\cosh 3\pi/2}+\cdots\right).$$

The present theorem is the special case when $n=0$.

Ramanujan's theorem was first proved by C. T. Preece, using contour integration (*Journal London Math. Society*, vol. 3, 1928, p. 215).

Solved also by H. S. Grant, H. Y. Hsü and S. F. Lee (jointly), J. P. McCarthy, C. D. Olds, and the Proposer.

Concurrent Lines in a Pentagon

4193 [1946, 160]. *Proposed by Hüseyin Demir, Columbia University*

If on the sides of an arbitrary pentagon $A_1A_2A_3A_4A_5$ the triangles $B_iA_{i+2}A_{i+3}$ (with indices reduced mod 5) are constructed such that $B_iA_{i+2} \| A_iA_{i+1}$, and $B_iA_{i+3} \| A_iA_{i+4}$, then the lines A_iB_i concur in a point C.

Solution by J. W. Clawson, Ursinus College, Collegeville, Pennsylvania. Take the triangle $A_1A_2A_4$ as the triangle of reference for a system of homogeneous trilinear coördinates. Let A_1 be (1, 0, 0), A_2 (0, 1, 0,), A_3 (d, e, f), A_4 (0, 0, 1), A_5 (k, l, m).

Then the equations of the line through A_3 parallel to A_1A_2 and of the line through A_4 parallel to A_1A_5 are

$$afx + bfy - (ad + be)z = 0, \qquad alx + (bl + cm)y = 0.$$

Thus B_1 has the coördinates

$$(bl + cm)(ad + be), \qquad -al(ad + be), \qquad acfm;$$

and the equation of A_1B_1 is

$$cfmy + l(ad + be)z = 0.$$

In the same way the equations of A_2B_2 and A_4B_4 are found to be

$$cfmx + d(ak + bl)z = 0, \qquad l(ad + be)x - d(ak + bl)y = 0.$$

These three lines are easily seen to be concurrent in a point C which has the coördinates

$$d(ak + bl), \qquad l(ad + be), \qquad -cfm.$$

Using the triangles $A_1A_3A_4$ and $A_2A_4A_5$ we can prove in the same way that A_3B_3 and A_5B_5 also pass through the point C.

Solved also by the Proposer.

Editorial Note. Clawson gave a second proof using the converse of Ceva's Theorem. The Proposer employed the pencils of lines A_3B_3, A_4B_4 formed when the side A_2A_3 rotates about A_3, other sides remaining fixed; since the correspondence between A_3B_3 and A_4B_4 is homographic, the locus of the intersection C of the rays A_3B_3, A_4B_4 is a conic; this conic decomposes into A_1B_1 and A_3A_4, thus giving the proof.

A Quadrilateral Similar to a Cyclic Quadrilateral

4194 [1946, 160]. *Proposed by R. Goormaghtigh, Bruges, Belgium*

In each of the triangles formed by three of the vertices of a cyclic quadrilat-

eral, we consider the projection of the orthocenter on the circumdiameter parallel to the Simson line of the fourth vertex of the quadrilateral with respect to the triangle. The four projections form a quadrilateral inversely similar to the one given and are on a circle concentric to the circumcircle of that quadrilateral.

I. *Solution by J. H. Butchart and Richard Meyer, Arizona State College at Flagstaff.* Let the vertices of the cyclic quadrilateral be A_1, A_2, A_3, A_4 and the orthocenter of the triangle formed by the omission of A_i be H_i. Let the projection of H_i of the line through the circumcenter O parallel to the Simson line of A_i be A_i'. We wish to show (1) that $\sphericalangle A_i'OA_j' = -\sphericalangle A_iOA_j$, and (2) that $OA_1' = OA_2' = OA_3' = OA_4'$.

It is well known that the Simson line of A_1 is parallel to A_2K, where K is the intersection with the circumcircle of the line through A_1 perpendicular to A_3A_4, and similarly the Simson line of A_2 is parallel to A_1L, where A_2L is perpendicular to A_3A_4 and L is on the circumcircle. Since A_1K and A_2L are parallel, the angle between A_2K and A_1L equals $\sphericalangle A_1OA_2$. For the proof of (2), note that OH_1 is the vector sum of OA_2, OA_3, OA_4 and that A_2K makes the same angles with these segments as A_1L makes with OA_1, OA_4, OA_3. The projections OA_1', OA_2' of OH_1, OH_2 on parallels to A_2K, A_1L through O are therefore equal. It is clear that the arc A_1A_2 is opposite in direction to the arc KL and thus that $\sphericalangle A_1'OA_2' = -\sphericalangle A_1OA_2$.

II. *Solution by Ou Li, Yenching University, Peiping, China.* In a system of complex coördinates having the given circumcircle as its base circle, the equation of the circumdiameter parallel to the Simson line of A_1 with respect to the triangle $A_2A_3A_4$ is

$$t_1x - t_2t_3t_4\bar{x} = 0,^*$$

where t_i are the coördinates of the vertices A_i $(i=1, 2, 3, 4)$. Then the coördinates of the projection H_1' of the orthocenter H_1 of the triangle $A_2A_3A_4$ on this line are easily found to be

$$x_1 = \tfrac{1}{2}\sigma_2\bar{t}_1 = \tfrac{1}{2}|\sigma_2|\,T\bar{t}_1$$

where $|T| = 1$ and $\sigma_2 = \sum t_it_j$. Analogous results evidently hold for the other three cases. Hence the four projections lie on a circle which is concentric to the base circle and whose radius is $\frac{1}{2}|\sigma_2|$, the radius of the base circle being taken as unity.

Furthermore, since

$$\frac{|x_1 - x_2|}{|t_1 - t_2|} = \frac{|\sigma_2|}{2},$$

the quadrilateral H_i' is similar to the given quadrilateral A_i. As x_it_i is a constant, the two quadrilaterals are situated in an inverse order. Also, from the relation

$$\frac{x_i - x_j}{\bar{x}_i - \bar{x}_j} = -t_st_k = \frac{t_s - t_k}{\bar{t}_s - \bar{t}_k} \qquad (i \neq j \neq s \neq k),$$

* R. Goormaghtigh, Analytic Treatment of Some Orthopole Theorems, this MONTHLY, vol. 46 1939 p. 266.

it follows that $H_i'H_j'$ is parallel to A_iA_j.

Solved also by J. W. Clawson and the Proposer.

Upper Bound for an Integral

4198 [1946, 225]. *Proposed by C. D. Olds, San Jose State College*

Prove that

$$\frac{1}{(n-1)!}\int_n^\infty w(t)e^{-t}dt < \left(\frac{2}{e}\right)^n,$$

where t is real, n is a positive integer, and $w(t) = (t-1)(t-2)\cdots(t-n+1)$.

Solution by Philip Davis, Harvard University, and A. M. Peiser, Rutgers University. Let

$$H_{n-1} = \frac{1}{(n-1)!}\int_n^\infty w(t)e^{-t}dt$$

and write $I_n = H_n e^{n+1}$. While a shorter proof of the stated inequality is possible, it seems desirable to establish the improved estimate

$$H_{n-1} < (e-1)^{-n}.$$

This is the best possible estimate. We shall show, in fact, that

$$H_{n-1} \sim (e-1)^{-n}, \qquad n \to \infty. \tag{2}$$

A simple change of variable yields

$$I_n = \frac{1}{n!}\int_0^\infty (t+1)(t+2)\cdots(t+n)e^{-t}dt.$$

Now,

$$\begin{aligned} I_{n+1} - I_n &= \frac{1}{(n+1)!}\int_0^\infty t(t+1)\cdots(t+n)e^{-t}dt \\ &= K_n + e^{-1}I_{n+1}, \end{aligned}$$

where

$$K_n = \frac{1}{(n+1)!}\int_0^1 t(t+1)\cdots(t+n)e^{-t}dt. \tag{3}$$

Thus

$$(1-e^{-1})I_{n+1} - I_n = K_n \tag{4}$$

and, in particular,

$$(1-e^{-1})I_{n+1} > I_n,$$

so that $J_n = I_n(1-e^{-1})^{n+1}$ is a monotonic increasing sequence. Using (4), we have

$$J_{n+1} - J_n = K_n(1 - e^{-1})^{n+1},$$

so that

$$J_m - J_1 = \sum_{n=1}^{m-1} (J_{n+1} - J_n) = \sum_{n=1}^{m-1} K_n(1 - e^{-1})^{n+1}.$$

It follows from (3) that $K_n < 1$, $n = 1, 2, \cdots$, so that we may write

$$\lim_{n\to\infty} J_n = J_1 + \sum_{n=1}^{\infty} K_n(1 - e^{-1})^{n+1}. \tag{5}$$

Now for $t \geqq 0$,

$$e^t = [1 - (1 - e^{-1})]^{-t} = 1 + t(1 - e^{-1}) + \frac{t(t+1)(1 - e^{-1})^2}{2!} + \cdots,$$

or

$$1 = e^{-t} + te^{-t}(1 - e^{-1}) + \frac{t(t+1)e^{-t}(1 - e^{-1})^2}{2!} + \cdots.$$

Integrating over $0 \leqq t \leqq 1$, a clearly justified termwise integration yields

$$1 = \int_0^1 e^{-t}dt + (1 - e^{-1}) \int_0^1 te^{-t}dt + \sum_{n=1}^{\infty} K_n(1 - e^{-1})^{n+1},$$

and by (5)

$$\lim_{n\to\infty} J_n = 1 + J_1 - \int_0^1 e^{-t}dt - (1 - e^{-1}) \int_0^1 te^{-t}dt$$

An easy computation shows that $J_1 = 2(1 - e^{-1})^2$, and that

$$\lim_{n\to\infty} J_n = 1.$$

This is the desired asymptotic result (2). The inequality (1) follows immediately from the monotonicity of J_n.

Also solved by D. W. Alling, Joshua Barlaz, P. T. Bateman and N. J. Fine, R. G. Blake, J. E. Brock and M. J. Gottlieb, Paul Brock, F. A. Butter, Jr., N. J. Fine, Fritz Herzog, P. M. Hummel, J. B. Kelly, S. F. Lee, A. S. Peters, M. W. Powell, J. H. Simester, J. G. Wendel, and the Proposer.

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All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Mathematics of Finance. By F. S. Harper. Scranton, Pa., International Textbook Co., 1946. 9+327 pages. $3.25.

In the author's words, "It is the purpose of this book to present the subject so that anyone with a background of college algebra may obtain in a one semester, or better a two-semester, course a working knowledge of the fundamentals of the subject which will enable him to understand and apply them to a wide range of practical problems." The book attains this goal in an admirable fashion. It is well written, carefully edited, and quite readable. In the development of the theory and the solutions of problems, emphasis is placed on the use of line diagrams, the equation of value principle, and a clear understanding of the origin and meaning of the tabulated functions. Considerable space is devoted to solutions of illustrative examples in which convenient methods of computing are described. The use of logarithms is encouraged throughout.

The book differs in many respects from others on the same subject, the most striking difference being in the order of the topics. The table of contents will indicate the unusual order. I. Interest rates. II. Ordinary annuities. III. Discount rates. IV. More general annuities. V. Bonds. VI. Miscellaneous problems. VII. Life annuities. VIII. Life-insurance premiums. IX. Terminal reserves.

Compound interest is introduced at the outset, simple interest being introduced later as an approximation to compound interest. The discussion of simple discount is postponed to Chapter III where it is defined as a convenient approximation to compound discount. This order results in the desirable repetition of certain fundamental ideas, as well as the early introduction of ordinary annuities.

The general annuity is treated in a simple manner by modifying the payment period to make it coincide with the interest-conversion period. Amortization, sinking-funds, capitalized cost, and depreciation are discussed adequately in Chapter VI. The treatment of life annuities and life insurance is particularly good. A feature of the text is the use of the 1937 Standard Annuity Table as well as the American Experience Table. Over fifty sets of exercises furnish numerous problems for drill. "Since numerical answers rarely give any clue to the method of solution to be employed, each exercise and problem is followed conveniently by its answer."

Tables at the end of the book have the standard functions tabulated on the left-hand pages and the corresponding logarithms on the right. The compound interest functions are given to seven places of decimals, the logarithms to six places. The tables are printed by offset from typescript, detracting somewhat

from the appearance of the book.

This reviewer found practically no misprints, the only one of importance being in the table on page 209. Here the formulas are numbered incorrectly, and $x+n$ in the fourth line should read $x+m$. In problem 20, page 87, the discount rate d is used before it is defined. If i is the rate per coupon period, then t should be replaced by t/t' in formulas 37–38 on page 140. In order that the statement at the top of page 141 be correct, P should be replaced by P_0 and the phrase "equation (37)" should be deleted. The author's definition of the "and accrued interest price" for bonds is not the customary definition, and no mention is made of the "professional practice" in connection with bonds. No distinction is made between a nominal rate *compounded* p times a year and a nominal rate *payable* p times a year.

To sum up, this book is a refreshing addition to the texts on the mathematics of finance. It should prove to be an effective teaching instrument.

H. D. LARSEN

NEW BOOKS RECEIVED

A First Course in Mathematical Statistics. By C. E. Weatherburn. Cambridge, at the University Press; New York, The Macmillan Company, 1946. 15+271 pages. $3.50.

Essentials of College Algebra and Mathematics of Investment. By W. L. Hart. Boston, D. C. Heath and Co., 1946. 10+304+126 pages. $4.75.

A Locus with 25920 Linear Self-transformations. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 39.) By H. F. Baker. Cambridge, at the University Press; New York, The Macmillan Company, 1947. 11+107 pages. $2.00.

Mathematical Recreations and Essays. By W. W. R. Ball. Revised by H. S. M. Coxeter. New York, The Macmillan Company, 1947. 16+418 pages. $2.95.

Methods of Mathematical Physics. By Harold Jeffreys and Bertha Jeffreys. Cambridge, at the University Press; New York, The Macmillan Company, 1946. 9+679 pages. $15.00.

The Methods of Plane Projective Geometry Based on the Use of General Homogeneous Coördinates. By E. A. Maxwell. Cambridge, at the University Press; New York, The Macmillan Company, 1946. 19+230 pages. $2.75.

The Cambridge Four-figure Mathematical Tables. New Edition. Cambridge, at the University Press; New York, The Macmillan Company, 1946. 32 pages. $0.45.

Plane Trigonometry. Revised Edition. By W. K. Morrill. New York, Rinehart and Co., 1946. 10+245 pages. $2.50.

The Theory of Functions of Real Variables. By L. M. Graves. New York and London, McGraw-Hill Book Co., 1946. 10+300 pages. $4.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

KAPPA MU EPSILON CONVENTION

The sixth national convention of *Kappa Mu Epsilon* was held at Illinois State Normal University at Normal, Illinois during April 10–12, 1947. One hundred seventy-one members of the fraternity representing twenty-six chapters were registered at the meetings.

The general theme, as announced by the chairman of the convention, Miss Elinor B. Flagg of *Illinois Alpha*, the host chapter, was Keep Mathematics Effective. This theme was evident at council meetings as well as at general sessions where the emphasis was placed on greater participation in mathematics by undergraduates. The National Council approved plans whereby *The Pentagon*, the official publication of *Kappa Mu Epsilon*, would publish more student papers. To encourage the writing of papers by undergraduates, a prize is to be given to the student author of the best paper published in *The Pentagon* in each biennium. This prize is to be an all-expense trip to the next national convention.

A committee was appointed to investigate and compile reports on opportunities for mathematicians in the various industries and professions together with the type of training needed for each. These reports are to be published periodically in the official journal.

The advisability of dividing *Kappa Mu Epsilon* into districts with separate district officers to make the work of *Kappa Mu Epsilon* more effective was also discussed. Definite action was postponed until the present growth of the fraternity became stabilized.

The convention approved applications for charters of five new chapters. These are:

William Jewell College at Liberty, Missouri
Texas State College for Women at Denton, Texas
Texas Christian University at Fort Worth, Texas
Baldwin-Wallace College at Berea, Ohio
Mount Mary College at Milwaukee, Wisconsin.

These chapters will receive their charters before the end of the semester. Other colleges and universities have indicated their interest in the fraternity but further action on these was postponed until September, 1947.

The principal address was given by Professor C. V. Newsom of Oberlin College, Oberlin, Ohio, Past-President of *Kappa Mu Epsilon*. The title of his paper was *The Mathematical Method*. Other faculty members who addressed the convention were:

President R. W. Fairchild of Illinois State Normal University
Professor E. R. Sleight, *Michigan Alpha*, National President of *Kappa Mu*

Epsilon and presiding officer at all meetings

Professor H. Van Egen of *Iowa Alpha*, President-Elect of *Kappa Mu Epsilon*

Miss Ruth Yates, Illinois State Normal University

Professor E. H. Taylor, *Illinois Beta*

Miss E. Marie Hove, *New York Alpha*, National Secretary of *Kappa Mu Epsilon*

Professor H. D. Larsen, *New Mexico Alpha*, Editor of *The Pentagon*

Sister Helen Sullivan, *Kansas Gamma*, National Historian of *Kappa Mu Epsilon*

Professor L. F. Ollmann, *New York Alpha*, National Treasurer of *Kappa Mu Epsilon*

Professor C. N. Mills of *Illinois Alpha*.

The following undergraduate papers were presented and will be published in *The Pentagon*:

Pattern forms of divisibility by Robert J. Weeks of *Illinois Alpha*

Computation of firing data for field artillery by Thomas Selby of *Michigan Beta*

A plea for non-isolationism in mathematics by Victoria Fritton, of *Kansas Gamma*

Mathematics in Scotland from 1717–1838 by Shirley Searls of *Michigan Alpha* and read by Audrey R. Schuett of *Michigan Alpha*.

Specially composed *Kappa Mu Epsilon* songs were sung by Victoria Fritton of *Kansas Gamma*, Miss B. Rohr of *New York Alpha* and Doris Wyatt of *Texas Beta*.

The officers elected for the biennium 1947–49 are:

President, Professor H. Van Engen, Iowa State Teachers College, Cedar Falls, Iowa.

Vice-President, Professor H. R. Mathias, Bowling Green State University, Bowling Green, Ohio.

Secretary, Miss E. Marie Hove, Hofstra College, Hempstead, New York.

Treasurer, Professor L. F. Ollmann, Hofstra College, Hempstead, New York.

Historian, Professor C. C. Richtmeyer, Central Michigan College of Education, Mount Pleasant, Michigan.

Immediate Past-president, Professor E. R. Sleight, Albion College, Albion, Michigan.

Editor of *The Pentagon*, Professor H. D. Larsen, University of New Mexico, Albuquerque, New Mexico.

Metropolitan Intercollegiate Mathematics Convention

The second annual Metropolitan Intercollegiate Mathematics Convention was sponsored by the *Pi Mu Epsilon* chapters of Brooklyn College, Hunter College, and New York University, on April 19, 1947 and held at Brooklyn College, Brooklyn, New York. The program included the following addresses:

Introductory address by George Shapiro of Brooklyn College

Topics in Topology by Professor Paul A. Smith of Columbia University

Applications of Probability by Edward C. Molina of Newark College of Engineering.

The committee members in charge of the convention were: George Shapiro, Brooklyn College; Gerard Washnitzer, Brooklyn College; Lillian Kaleko, Brooklyn College, Secretary; Cecile Salwen, Hunter College; Annette Drucker, Hunter College; Joyce Marrits, Hunter College; Elaine Margolin, New York University; and Bernice Goldberg, New York University.

CLUB REPORTS

Mathematics Club, University of Richmond

The first meeting of the current session was devoted to the initiation and orientation of new members. Subsequently, two meetings were held at which the topics listed below were presented. Refreshments were served at each meeting.

An introduction to the theory of relativity by Dr. B. C. Holtzclaw, Professor of Philosophy, University of Richmond

Additions to the periodic table by Dr. W. E. Trout, Jr., Professor of Chemistry, University of Richmond.

Kappa Mu Epsilon, Wayne University

The Wayne University *Mathematics Club* was formally installed as the *Michigan Gamma* chapter of *Kappa Mu Epsilon* on May 10, 1946. The ceremony was conducted in the Rackham Building while the installation banquet, honoring the twenty-seven charter members, was held at the Book-Cadillac Hotel. Professor E. R. Sleight of Albion College, National President of *Kappa Mu Epsilon*, officiated at the installation of the new chapter and gave the banquet address.

Topics discussed during subsequent meetings include:

Two mathematical puzzles, by Ted Slaby

Comprehension and retention, by Nadine Zelinek

Curves of constant breadth, by Professor John W. Baldwin.

The officers include: President, Ted Slaby; Corresponding Secretary, Professor D. C. Morrow.

Kappa Mu Epsilon, Illinois State Normal University

Illinois Alpha chapter of *Kappa Mu Epsilon* held eleven meetings during 1945–46, of which two were special meetings. Six new members were added to give a total membership of thirty-one for the year. Some of the activities were: a news letter was sent to all *Illinois Alpha* alumni, a home-coming breakfast for the alumni, and a Spring picnic.

At the annual Spring banquet, at which forty-three members were present, Miss Eunice Blackburn, missionary in Yucatan, spoke on *Education in Yucatan*.

Other outstanding programs include:

Exhibit and discussion of old mathematics books by Dr. C. N. Mills

Significant figures by Dr. C. T. McCormick

Introduction to the mathematics of factor analysis by Dr. B. R. Ullsvick.

Officers for 1945–46 were: President, Maxine Sponsler; Vice-President, Mary Donnell; Secretary, Martha Lewis; Treasurer, LaVerne Wenzelman; Historian, Janice Posey; Social Director, Alice McCorkle; Corresponding Secretary, Dr. C. N. Mills; Sponsor, Miss Elinor B. Flagg.

NEWS AND NOTICES

EDITED BY B. W. JONES, Cornell University

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

SUMMER COURSES

The University of Notre Dame announces the following advanced courses for its summer session from June 18 to August 13: by Professor Ross, introduction to modern algebra; by Professor Vandiver (University of Texas), advanced topics in algebra and number theory; by an instructor yet to be announced, introduction to topology.

Syracuse University announces the following advanced courses from June 16 to September 7: by Professor Bers, differential geometry; by Dr. Bruns, vector analysis; by Dr. Dubisch, higher algebra; by Professor Gelbart, functions of a real variable; by Dr. Gilbert, introduction to topology; by Professor Harwood, differential equations; by Professor Loewner, fluid dynamics; and by Professor Welch, probability. From July 7 to August 16 the following courses will be offered: by Dr. Bernstein, history of mathematics; by Professor Cairns, fundamentals of analysis; and by Professor Stokes, introduction to modern mathematics.

DUKE UNIVERSITY INSTITUTE FOR TEACHERS

The Duke University Institute for Teachers of Mathematics will be held from August 5 to August 15. This conference, under the direction of W. W. Rankin, is the seventh annual session of the Mathematics Institute for Teachers. In the past six summers more than 450 teachers from twenty states have attended the Institute. The theme of this year's Institute will be "Mathematics at Work."

The purpose of the Institute is to bring together high school and college teachers of mathematics, to study intensively the problems of common interest and to learn new uses of mathematics. The work of the Institute centers around the Mathematics Laboratory located in the West Duke Building on the East Campus. The Mathematics Laboratory makes available in one place a wide range of materials relating mathematics to science, industry, engineering, education, and commerce.

There will be a registration fee for the Institute of $3.00. This fee includes admission to the recreation facilities, tennis, swimming, *etc.*, of the Duke University Summer Session. Room and board will be furnished by Duke University at the rate of $3.00 for a double room and $3.50 per day for a single room. Room reservations should be made early. No fee or deposit need be made before arriving. Further information may be obtained by writing to Professor Rankin.

AID TO DEVASTATED LIBRARIES

The Committee on Aid to Devastated Libraries has prepared for shipment abroad a number of packages containing periodicals and a few books. These packages are being sent to libraries in Belgium, China, Czechoslovakia, Finland, Germany, Greece, Hungary, India, Italy, and The Netherlands, through the assistance of the American Book Center. The Committee wishes to thank all those members of the Association who have made the sending of these packages possible by their response to the appeal which was published in the MONTHLY for October, 1946. It also wishes to express its indebtedness to the American Book Center (Library of Congress, Washington 25, D. C.), which continues throughout the present year the important activity of collecting and forwarding all material that can be used in the intellectual reconstruction abroad.

The amount and the character of the material that the Committee has received thus far has fallen far short, however, of what is needed to meet the needs that have been made known. There are needed many books published during the war years, as well as reprints of important papers and journals. That is the reason for the present appeal. Help can be given as follows:

(1) By contributions of money to be sent to the New York office of the Society. Checks should be made out to the American Mathematical Society (with an indication FOR FOREIGN LIBRARIES). Money so contributed will be used solely for the purchase of books and journals not published by either the Society or the Association, and not available to the Committee from other sources. The cost of preparing the material for shipment, and of sending it abroad is defrayed by a small appropriation made for this purpose by the Trustees of the Society and by the cooperation of the American Book Center.

(2) By sending books and journals, in prepaid packages, to the American Mathematical Society, Butler Library, Columbia University, 531 West 116th Street, New York 27, N. Y., with the notation FOR FOREIGN LIBRARIES. In order to give an idea of the character of the books that are needed, an indication is given below of the specific requests that have been made. These requests have come from libraries of universities and scientific societies in Austria, Belgium, China, Czechoslovakia, England, Finland, France, Germany, Greece, Hungary, India, Italy, Japan, the Netherlands, Poland, Roumania, Yugoslavia.

There are several requests for journals published during the war years. There is particular need for the *Annals*, the *Duke Journal*, the *Journal of Mathematics and Physics*, as well as for the various journals published or supported by the

Society. Among the books asked for, the various numbers of the *Colloquium Series*, and the two numbers of *Mathematical Surveys* appear repeatedly; so do the volumes in the *Princeton Mathematical Series* and the *Annals of Mathematics Studies*. The following individual titles are taken from long lists of requests:

Albert, *Introduction to Algebraic Theories*,
Artin, *Galois Theory*
Bennett and Bayliss, *Formal Logic*
Birkhoff and MacLane, *Survey of Modern Algebra*
Churchill, *Fourier Series*
Davis, *Principles of Econometrics*
Dickson, *History of the Theory of Numbers*
Eisenhart, *Non-Riemannian Geometry*
Eisenhart, *Differential Geometry*
James, *Mathematics Dictionary*
Lane, *Treatise on Projective Differential Geometry*
Sternberg-Smith, *Theory of the Potential*
Tarski, *Introduction to Logic*
Taylor, *Vector Analysis*
Uspensky-Heaslet, *Elementary Number Theory*
Veblen and Young, *Projective Geometry*
Watson, *Bessel Functions*
Widder, *The Laplace Transform*
Wilks, *Mathematical Statistics*

LEO ZIPPIN
PAUL A. SMITH
ARNOLD DRESDEN, *Chairman*

PERSONAL ITEMS

Professor H. J. Ettlinger of the University of Texas represented the Mathematical Association at the installation of William V. Houston as President of Rice Institute on April 10, 1947.

Professor J. W. Alexander of the Institute for Advanced Study has been awarded an honorary doctorate of science by Princeton University.

Dr. A. S. Householder of the Monsanto Chemical Company has received a Bureau of Ordnance Development award for work with project N-111; Applied Psychology Panel.

Professor J. R. Kline of the University of Pennsylvania has been elected to membership in the Society of Science and Letters at Warsaw.

K. A. Bush of Mohawk College has been promoted to an associate professorship.

F. M. Carpenter of the State University of Iowa has been appointed to an assistant professorship at Missouri School of Mines, Rolla, Missouri.

Associate Professor H. S. M. Coxeter of the University of Toronto has been appointed to a visiting professorship at the University of Notre Dame.

Assistant Professor M. J. Gottlieb of Washington University is on leave of absence this term to work at the Institute for Advanced Study.

Drs. G. H. A. Grosheide and J. Haantjes of the Free University of Amster-

dam have been promoted to professorships. This is a correction of an item in the February issue of the MONTHLY.

Dr. F. C. Jonah of Chance Vought Aircraft Division of the United Aircraft Corporation has been promoted to the position of Staff Project Engineer.

C. W. Jordan, Jr., has been appointed to an assistant professorship at Williams College.

Dr. H. D. Kloosterman of the University of Leiden has been promoted to a professorship.

R. R. Kuebler of Dickinson College has been promoted to an assistant professorship.

Assistant Professor J. K. L. MacDonald of Cooper Union has been appointed to a professorship of graduate mathematical physics at New York University.

Assistant Professor M. G. Moore of Bradley University, Peoria, Illinois, has been promoted to an associate professorship.

Dr. Z. I. Mosesson of the Prudential Insurance Company of America has been promoted to the position of Assistant Mathematician.

Dr. W. D. Munro of the University of Minnesota has been promoted to an assistant professorship in mathematics and mechanics.

Dr. P. F. Nemenyi has been appointed physicist with the Naval Ordnance Laboratory, White Oak, Maryland.

Dr. J. Popken has been appointed to a professorship at the University of Utrecht.

Professor E. B. Wilson of the School of Public Health, Harvard University, has retired with the title emeritus.

The following appointments to instructorships are announced:
New York Institute of Optics: L. D. Levine
United States Naval Academy: J. R. Gorman

Word has been received of the death of Professor Ettore Bortolotti of the University of Bologna on February 17, 1947.

Professor Emeritus B. L. Remick of Kansas State College died March 18, 1947.

Professor W. T. Short of Oklahoma Baptist University died February 19, 1947.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

ANNUAL MEETING OF THE ILLINOIS SECTION

The twenty-fifth annual meeting of the Illinois Section of the Mathematical Association of America was held at Illinois State Normal University, Normal, Illinois, on Friday and Saturday, May 10–11, 1946.

There were forty-six persons in attendance, including the following twenty-

one members of the Association: Edith I. Atkin, D. R. Bey, S. F. Bibb, W. H. Coulter, Elinor B. Flagg, L. R. Ford, A. E. Gault, Mildred Hunt, E. C. Kiefer, W. C. Krathwohl, Luise Lange, C. T. McCormick, W. C. McDaniel, C. N. Mills, M. G. Moore, E. J. Moulton, E. W. Ploenges, J. M. Sachs, M. Anice Seybold, E. H. Taylor, B. R. Ullsvick.

At the business meeting the following officers were elected for the coming year: Chairman, C. N. Mills, Illinois State Normal University; Vice-Chairman, M. G. Moore, Bradley Polytechnic Institute; Secretary, E. C. Kiefer, James Millikin University. It was decided to hold the next meeting on May 10–11, 1947 at Bradley Polytechnic Institute in Peoria, Illinois.

The following papers were presented:

1. *A note on the history of the concept of infinity in mathematics*, by Dr. E. H. Taylor, Eastern Illinois State Teachers College.

The speaker noted the central role of the concept of infinity in the development of mathematics. He discussed the place of the concept in the growth of modern geometry, in the development of the calculus, and in modern attempts to attain rigor in mathematics. Attention was called to the conflict between the school of Hilbert and that of Brouwer growing out of difficulties with infinite processes and infinite classes.

2. *On a graphical interpretation of the criteria for conic sections*, by Dr. Luise Lange, Chicago City Junior Colleges, Woodrow Wilson Branch.

3. *Forum: What college mathematics for the returning service man?* by Professor C. N. Mills, Illinois Normal University, Professor E. C. Kiefer, James Millikin University, and Professor S. F. Bibb, Illinois Institute of Technology.

The various speakers told of efforts made in their own schools to assist returned service men.

4. *Two theorems on Brocard points derived from group theory*, by Sister Mary Phillip, Rosary College.

A geometric representation of the dihedral group G_6 in the real inversive plane shows that certain sets of points with special significance from the group point of view play an important part in the geometry of the configuration. The triangles of a certain poristic system under G_6 have the same Brocard points, namely, the transforms of their common circumcenter under the invariant subgroup of G_6. Moreover, the Brocard angles of these triangles are equal.

The Brocard angle of this system of triangles under G_6 is equal to the Brocard angle which determines the circumcircle of the system as a Neuberg circle on a segment of the Desargues' axis of related pairs of triangles of the system with limiting points the common isodynamic points of the triangles of the system.

5. *Application of mathematics to war problems*, by Professor E. J. Moulton, Northwestern University.

Professor Moulton talked briefly of the extent to which professional mathe-

maticians participated in the war effort. Dividing the field of mathematics into nine sections ranging from arithmetic to topology, he indicated where each was used in connection with the recent conflict. He spoke of such diverse problems as the design of clothing, weather forecasting, and the atomic bomb. Numerous other applications were mentioned.

E. C. KIEFER, *Secretary*

NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pennsylvania, on Saturday, November 30, 1946. Professor P. A. Caris, Chairman of the Section, presided at the morning and afternoon sessions.

There were fifty-one present, including the following thirty-five members of the Association: C. B. Allendoerfer, Joshua Barlaz, P. T. Bateman, A. L. Billig, T. A. Botts, H. W. Brinkmann, L. H. Bunyan, W. B. Campbell, P. A. Caris, J. A. Clarkson, J. W. Clawson, L. J. Deck, F. L. Dennis, Arnold Dresden, W. H. Gottschalk, J. F. Heyda, J. R. Kline, F. L. Manning, Clifford Marburger, D. L. McDonough, S. S. McNeary, A. E. Meder, Jr., Martin Moliver, W. R. Murray, C. A. Nelson, J. C. Oxtoby, C. J. Rees, I. J. Schoenberg, Benjamin Slepin, L. L. Smail, E. P. Starke, G. L. Walker, R. M. Walter, G. C. Webber, Anna Pell Wheeler.

At the business meeting the following officers were elected for the coming year: Chairman, C. J. Rees, University of Delaware; Secretary, W. H. Gottschalk, University of Pennsylvania. The Program Committee for the next meeting will be: I. J. Schoenberg (Chairman), University of Pennsylvania, C. B. Allendoerfer, Haverford College, and A. E. Pitcher, Lehigh University. The next meeting will be held on Saturday, November 29, 1947.

The program consisted of the following papers:

1. *Convex sets*, by Professor T. A. Botts, University of Delaware.

The elementary properties of convex sets in the euclidean plane were discussed from the intuitive geometric viewpoint. The plane-of-support property of convex sets was established in two (well known) ways, each of them avoiding the type of limiting process usually employed. It was pointed out that the geometric arguments used could be phrased analytically so as to be valid for the corresponding theorems in n-dimensional euclidean space.

2. *Slope in solid analytic geometry*, by Professor C. B. Allendoerfer, Haverford College.

This paper has been published in this MONTHLY, vol. 53 (1946), pp. 241–247.

3. *Generalizations of the Weierstrass approximation theorem*, by Professor Edwin Hewitt, Bryn Mawr College, introduced by Professor W. R. Murray.

The celebrated approximation theorem of Weierstrass asserts that, given a closed interval $[a, b]$ in the real number system, a real-valued continuous function $\phi(x)$ defined on $[a, b]$, and any positive number ϵ, there exists a polynomial

$p(x)$ such that $|p(x)-\phi(x)|<\epsilon$ for all x in $[a, b]$. Many different proofs have been given for this theorem, some depending upon the existence of integrals, others using uniform continuity. M. H. Stone has proved (*Trans. Amer. Math. Soc.*, vol. 41, 1937, p. 466, Theorem 82) a striking generalization of this theorem, as follows. Let X be any bicompact Hausdorff space, and $\mathfrak{F}$ any set of real-valued continuous functions defined on X such that for every $p, q \epsilon X$ which are distinct points, there is a function $f \epsilon \mathfrak{F}$ such that $f(p) \neq f(q)$. Then any continuous real-valued function defined on X can be approximated uniformly to an arbitrary degree of accuracy of means of polynomials $p(f_1, \cdots, f_n)$, where $f_1, \cdots, f_n$ are in $\mathfrak{F}$. This theorem is proved as a special case of a more general approximation theorem valid in arbitrary completely regular spaces.

W. H. GOTTSCHALK, *Secretary*

FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the Johns Hopkins University on Saturday, December 7, 1946, with a morning session, luncheon, and an afternoon session. Professor W. K. Morrill, Chairman of the Section, presided at both sessions.

The attendance was one hundred and one including the following thirty-seven members: N. H. Ball, Archie Blake, S. G. Bourne, H. H. Campaigne, Abraham Cohen, G. F. Cramer, J. A. Duerksen, E. J. Finan, M. K. Fort, Jr., C. H. Frick, Michael Goldberg, J. R. Hammond, G. A. Hedlund, Sister Mary Cordia Karl, L. M. Kells, Katharine B. Keppler, V. L. Klee, Jr., W. D. Lambert, M. H. Martin, E. S. Mayer, Carol V. McCamman, Mary E. Meade, Emanuel Mehr, Joseph Milkman, T. W. Moore, F. D. Murnaghan, W. H. Norris, Jr., Grace S. Quinn, C. E. Rhodes, R. E. Root, E. D. Schell, H. R. Smith, S. Helen Taylor, Marian M. Torrey, W. R. Utz, A. L. Whiteman, G. T. Whyburn.

The first five of the following papers were read at the morning session. Professor Murnaghan's paper was read at the afternoon session at the invitation of the Section.

1. *Needed changes in curriculum materials and in methods as suggested by war-time experience*, by Dr. S. Helen Taylor, State Teachers College, Frostburg, Maryland.

As a result of her war-time experience, Dr. Taylor recommended several changes in the curriculum material now generally used in the teaching of mathematics at the college level.

2. *On systems of constructible number theory*, by Dr. David Nelson, George Washington University, introduced by the Secretary.

A formal system of classical number theory C may be extended by the addition of further logical primitives for implication, alternation, and the existential quantifier and further rules of inference to give an intuitionistic system I. This system may be further extended by the addition of a new primitive for negation to give a system N. Each of these extensions may be characterized by a depend-

ence among the logical symbols in C and in I. If the logical symbols of N are independent, an interpretation of the logical connectives which is divergent from the three represented by these systems is suggested; if not, a characterization of N is provided.

3. *A construction of the extended real number system*, by M. K. Fort, Jr., University of Virginia.

The terms "closed interval of rational numbers" and "directed set of closed intervals" were first defined. An "extended real number" was then defined to be a maximal directed set. The collection of all such extended real numbers contains in addition to the usual real numbers two elements which are called $-\infty$ and ∞. Finally, multiplication and addition were defined and some properties of the extended real number system were discussed.

4. *On the equation* $\phi(x)=n$, by V. L. Klee, Jr., University of Virginia.

A brief survey of previous results on the problem of obtaining solutions of $\phi(x)=n$ was followed by an outline of the proof and an application of the following theorem due to the author: If m is an odd integer greater than 1, expressible in the form

$$m = \prod (2^{j_i} + 1)^{a_i} \prod d_i(2^{k_i}d_i + 1)^{b_i}$$

where the integers $2^{j_i}+1$ and $2^{k_i}d_i+1$ are prime, if the a_i's and the b_i's are non-negative integers, and if

$$1 \leqq \sum j_i + \sum k_i \leqq n,$$

then

$$x = 2^{n+1} - \sum j_i - \sum k_i \prod (2^{j_i} + 1)^{a_i+1} \prod (2^{k_i}d_i + 1)^{b_i+1}$$

is a solution of the equation $\phi(x)=2^n m$. And if $\sum j_i+\sum k_i=n$, then $x/2$ is also a solution. Furthermore, all solutions are obtainable in this manner from expressions of m in the above form.

5. *On the decomposition of meromorphic functions*, by W. R. Utz, University of Virginia.

In a recent paper L. H. Loomis has given a necessary and sufficient condition for the decomposition of a meromorphic function, defined on the interior of the unit circle, into a regular, bounded, univalent function followed by a rational function. This condition is given in terms of the behavior of the function near the boundary of the region of its definition. In the present paper the author employs extensions to the boundary of mappings from the unit circle upon an arbitrary bounded simply connected region with locally connected boundary to secure an analogous theorem for a function meromorphic on a bounded, simply connected region with locally connected boundary.

6. *The teaching of mathematics below the college level*, by Professor E. D. Murnaghan, Johns Hopkins University.

This paper will be published in the *Mathematics Teacher*.

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Twenty-ninth Summer Meeting, New Haven, Connecticut, September 1–2, 1947.

Thirty-first Annual Meeting, Athens, Georgia, January 1, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

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ILLINOIS
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NORTHERN CALIFORNIA, Berkeley, January 24, 1948
OHIO
OKLAHOMA
PACIFIC NORTHWEST, Eugene, Oregon, March, 1948
PHILADELPHIA, Bryn Mawr, Pa., November 29, 1947
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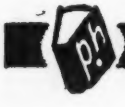

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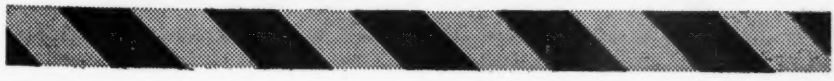

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AUGUST–SEPTEMBER 1947

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